Shear Lag Model for Regularly Staggered Short Fuzzy Fiber Reinforced Composite

In this article, we investigate the stress transfer characteristics of a novel hybrid hierarchical nanocomposite in which the regularly staggered short fuzzy fibers are interlaced in the polymer matrix. The advanced fiber augmented with carbon nanotubes (CNTs) on its circumferential surface is known as “fuzzy fiber.” A three-phase shear lag model is developed to analyze the stress transfer characteristics of the short fuzzy fiber reinforced composite (SFFRC) incorporating the staggering effect of the adjacent representative volume elements (RVEs). The effect of the variation of the axial and lateral spacing between the adjacent staggered RVEs in the polymer matrix on the load transfer characteristics of the SFFRC is investigated. The present shear lag model also accounts for the application of the radial loads on the RVE and the radial as well as the axial deformations of the different orthotropic constituent phases of the SFFRC. Our study reveals that the existence of the non-negligible shear tractions along the length of the RVE of the SFFRC plays a significant role in the stress transfer characteristics and cannot be neglected. Reductions in the maximum values of the axial stress in the carbon fiber and the interfacial shear stress along its length become more pronounced in the presence of the externally applied radial loads on the RVE. The results from the newly developed analytical shear lag model are validated with the finite element (FE) shear lag simulations and found to be in good agreement. [DOI: 10.1115/1.4027801]

Keywords: shear lag model, stress transfer, staggered, fuzzy fibers, hybrid, hierarchical, nanocomposite

1 Introduction

CNTs [1] have been emerged as the ideal candidates for multifarious applications due to their remarkable elastic and physical properties. A CNT can be viewed as a hollow seamless cylinder formed by rolling a graphene sheet. A great number of experimental and analytical studies have been carried out to estimate the elastic properties of CNTs [2–8], and reported that the axial Young’s modulus of CNTs is in the TeraPascal range. The quest for utilizing such exceptional elastic properties of CNTs has led to the opening of an emerging area of research concerned with the development of the two-phase CNT-reinforced nanocomposites [8–15]. However, the addition of CNTs in polymer matrix does not always result in improved effective properties of the two-phase nanocomposites. Several important factors, such as agglomeration, aggregation, and waviness of CNTs, and difficulty in manufacturing also play a significant role [16,17]. These difficulties can be alleviated by using CNTs as secondary reinforcements in a three-phase CNT-reinforced composite. In this case, CNTs are grown on the circumferential surfaces of the fiber reinforcements. For the first time, Thostenson et al. [18] synthesized CNTs on the circumferential surfaces of the carbon fibers using chemical vapor deposition (CVD) technique and found that the presence of CNTs at the fiber/matrix interface improves the interfacial shear strength of the hybrid nanocomposite. Using dc plasma-enhanced CVD technique, the growth of vertically aligned and well-separated array on an electrically conductive carbon microfiber has been demonstrated by Chen et al. [19]. Veedu et al. [20] fabricated the multifunctional composite in which the vertical arrays of CNTs are grown on the circumferential surfaces of the fibers. Their study revealed that the vertical arrays of CNTs in the thickness direction of the composite improve the multifunctional properties without compromising the in-plane properties, and also alleviate the problem of agglomeration of CNTs. Woven carbon fiber laminae have been functionalized by Kepple et al. [21] in situ by growing CNTs on the circumferential surface of the carbon fiber. In their work, growing of CNTs on the circumferential surfaces of the carbon fibers improved the fracture toughness of the cured composite by as much as 50% without any loss in the structural stiffness of the composite. Sager et al. [22] grew CNTs on the circumferential surface of T650 carbon fiber in an epoxy matrix by using thermal CVD, and reported that the interfacial shear strength of the resulting composite improves with the addition of a CNT coating. They attributed this improvement to the increase in the interphase yield strength as well as an improvement in the interfacial adhesion as a result of the presence of CNTs. CNTs reinforce the polymer matrix between the fibers by providing enhanced strength and toughness, as depicted in Figs. 1(a) and 1(b) [23,24]. The resulting laminated structure is described as a hybrid advanced composite laminate, rather than as a nanocomposite, and is perhaps best described as the FFRC [23,25]. Recently, the effective thermoelastic properties of such FFRC have been estimated by Chatzigeorgiou et al. [26] and Kundalwal and Ray [27–30].

The transfer of load from the surrounding matrix to the fiber is one of the fundamental micromechanical processes determining the composite strength. It is a complex process that depends on the fiber/matrix interfacial properties, constitutive behavior of matrix, geometrical arrangement of the fibers, their volume fraction, and strength of the fibers [31]. However, considerable simplification comes from considering the shear lag load transfer models. Previous shear lag studies have reported that the reinforced CNTs in the nanocomposites significantly enhance the stress transfer characteristics of the CNT-reinforced composites [32–37]. Ray et al. [38] carried out the load transfer analysis of the short carbon fiber reinforced composite in which aligned CNTs are radially
grown on the circumferential surfaces of the carbon fibers. In their study, the short carbon fibers being coated with the radially grown CNTs were assumed to touch each other laterally and hence, a two-phase shear lag model was developed. In practice, CNT-coated carbon fibers may not touch each other laterally and the resulting composite will be composed of three phases; namely, the carbon fiber, the CNT-reinforced polymer matrix nanocomposite (PMNC), and the polymer matrix. Pavia and Curtin [39] developed a shear lag model for a “fuzzy fiber” ceramic matrix composite containing wavy, finite length nanofibers having a statistical distribution of strengths as a function of all the material parameters, including morphology.

Unidirectional nanocomposite structures with parallel staggered platelet reinforcements are widely observed in the natural biological materials. Previous studies indicated that the staggered micro/nanostructures in the natural biological materials play a crucial role in their superior mechanical properties [40–45]. To understand the principle of the structural hierarchy in load-bearing biological materials, Zhang et al. [46] developed a quasi-self-similar structure model to show that depending on the mineral content, there exists an optimal level of structural hierarchy for maximum toughness of hierarchical materials. Bar-On and Wagner [47] derived analytical shear lag model for predicting the mechanical behavior of the staggered platelet-matrix structures. They compared their analytical results with FE simulations and found good agreement between the two for a wide range of structural and material parameters. Recently, Lei et al. [48] investigated the effects of nonuniform or random distributions of unidirectional short fibers on the mechanical properties of two-phase composites considering the isotropic constituent phases. They indicated that the stairwise and regular staggering distributions of fibers improve the overall mechanical performance of the composite, such as the stiffness and the strength when compared with those of the reinforcing fibers. They further observed large failure strain and energy storage capacity comparable with those of the soft matrix.

However, existing shear lag studies of CNT-reinforced nanocomposites do not account for such staggering effect for the adjacent RVEs and cannot provide accurate description of the overall load transfer characteristics of the CNT-reinforced composites. For example, Ray and Kundalwal [49,50] and Kundalwal and Ray [51] carried out the thermomechanical shear lag analysis of a novel hybrid hierarchical SFFRC considering zero shear tractions along the length of the RVE and neglecting the so-called staggering effect. The schematic diagram shown in Fig. 2(a) represents the SFFRC lamina in which half overlapping short fuzzy fibers are embedded in the polymer matrix. In their study, a three-phase shear lag model is developed for this novel hybrid nanocomposite considering zero shear tractions along the length of the RVE and neglecting the staggering effect. However, these earlier shear lag studies do not account for the staggering effect and cannot provide accurate description of the overall load transfer characteristics of hybrid hierarchical nanocomposites. In order to analyze the complex load transfer characteristics of hybrid hierarchical nanocomposites, higher level of shear lag model must be developed considering the staggering effect and three-dimensional loads. However, to the best of the authors’ knowledge, existing shear lag studies on the hybrid nanocomposites disregard the staggering effect. Such effects arise as a result of the interactions between adjacent staggered RVEs, and become especially significant in tightly packed architectures. This is indeed the motivation behind the current study. The present study is devoted to the development of a comprehensive three-phase shear lag model for analyzing the stress transfer characteristics of hybrid hierarchical CNT-reinforced nanocomposite incorporating the staggering effect.

The remainder of this article is organized as follows: Section 2 briefly presents the architecture of the hierarchical SFFRC. In Sec. 3, the development of the Mori–Tanaka (MT) model is presented which provides the effective elastic properties of the
PMNC phase as input for the shear lag analysis. Development of a three-phase shear lag model is presented in Sec. 4. In Sec. 5, analytical results are validated with the experimental results and the FE simulations, and are used to illustrate the load transfer characteristics of the SFFRC. In Sec. 6, main inferences are drawn from the present study. In the Appendix, constant coefficients are presented which are obtained in the course of deriving the shear lag model in Sec. 4.

2 Architecture of a Novel Hierarchical SFFRC

Figure 2(a) represents a lamina of the SFFRC in which the parallel staggered short fuzzy fibers are reinforced in the polymer matrix and its in-plane cross section is shown in Fig. 2(b). A conceptual illustration of a fuzzy fiber is also shown in Fig. 3(a). When this fuzzy fiber is embedded into a polymer, the spaces between CNTs are filled up with the polymer and the radially aligned CNTs eventually reinforce the polymer matrix surrounding the carbon fiber along the direction transverse to the length of the carbon fiber. Thus, the augmented fuzzy fiber can be viewed as a circular cylindrical short composite fuzzy fiber (SCFF) in which the carbon fiber is embedded in the PMNC. In turn allows one to treat the SFFRC as a composite in which the SCFFs are the reinforcements being embedded in the polymer matrix as shown in Fig. 3(b). This figure also illustrates the 3D structure of the RVE-A under consideration, subject to the appropriate boundary conditions.

3 MT Model

MT model [52] is an Eshelby-type model which accounts for the interactions among the neighboring reinforcements. Due to its simplicity, the MT model has been reported to be the efficient analytical model for predicting the effective orthotropic elastic properties of the composites. Hence, a three-phase MT model is used to determine the effective orthotropic elastic properties of the PMNC which are required as input to the shear lag model development presented in Sec. 4. From the constructional feature of the SCFF, it may be viewed that the carbon fiber is wrapped by a lamina of the PMNC material. Such an unwound lamina of the PMNC is reinforced by CNTs along its thickness direction as shown in Fig. 4. The average effective elastic properties of the PMNC material surrounding the carbon fiber may be approximated by estimating the effective elastic properties of this unwound PMNC lamina. The effective elastic properties of the PMNC are estimated in the presence of an interphase between a CNT and the polymer matrix. Such an interphase models the non-bonded van der Waals interaction between a CNT and the polymer matrix [8,11,12,53–55]. The effective elastic properties of such interphase resembling a solid continuum can be determined by the molecular dynamics simulation and are readily available in open literature [8]. Employing the procedure of the MT model for the multiple inclusions [56], a three-phase MT model can be derived for the unwound PMNC. The explicit formulation of such three-phase MT model can be derived as

\[
[C^{\text{PMNC}}] = v_p[C^0][I] + v_n[C^0][A_n] + v_{\text{in}}[C^\text{in}][A_{\text{in}}] \left[ v_p[I] + v_n[A_n] + v_{\text{in}}[A_{\text{in}}] \right]^{-1}
\]

In Eq. (1), the respective superscripts n, in, and p denote the CNT fiber, the effective interphase, and the monolithic polymer matrix; whereas \(v_n, v_{\text{in}}, \) and \(v_p\) represent the volume fractions of the CNT, the interphase, and the polymer matrix, respectively, with respect to the RVE of the PMNC. The concentration tensors \([A_n]\) and \([A_{\text{in}}]\) appearing in Eq. (1) are given by

\[
[A_n] = \left[ [I] + [S_n] \right]^{-1} \left( (C^0) - [C^0] \right)
\]

\[
[A_{\text{in}}] = \left[ [I] + [S_{\text{in}}] \right]^{-1} \left( (C^\text{in}) - [C^\text{in}] \right)
\]

Furthermore, in the above matrices, \([S_n]\) and \([S_{\text{in}}]\) indicate the Eshelby tensors for the domains n and in, respectively, and \([I]\) is an identity matrix. Cylindrical molecular structure of a CNT may be treated as an equivalent solid cylindrical fiber [8,11,32]. Thus, the specific form of the Eshelby tensor for the cylindrical inclusion given by Qiu and Weng [57] is utilized to compute the matrices \([S_n]\) and \([S_{\text{in}}]\).

It may be noted that the elastic coefficient matrix \([C^{\text{PMNC}}]\) directly provides values of the effective elastic properties at a point in the portion of the PMNC material surrounding the carbon fiber, where the CNT is aligned with the 3-axis of the SFFRC. However, if we...
consider the local coordinate system \((1'-2'-3')\), as shown in Figs. 4 and 5(a), the matrix \([C_{nc}^{\text{eff}}]\) also provides the effective elastic properties at a point located in the PMNC, where the CNT axis \((3'-\text{axis})\) is oriented at an angle \(\theta\) with the 3-axis in the 2–3 plane. Thus, at any point in the PMNC surrounding the carbon fiber, the location dependent effective elastic coefficient matrix \([C_{nc}^{\text{eff}}]\) of the PMNC with respect to the principal coordinate system \((1–2–3)\) can be obtained by the following transformations:

\[
[C_{nc}^{\text{eff}}] = [T]^{-1}[C_{nc}][T]^{-1}
\]  

(4)

where

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & m^2 & n^2 & mn & 0 \\
0 & n^2 & m^2 & -mn & 0 \\
0 & -2mn & 2mn & m^2 - n^2 & 0 \\
0 & 0 & 0 & 0 & m - n \\
0 & 0 & 0 & 0 & n \\
\end{bmatrix}
\]

with

\[
m = \cos \theta \quad \text{and} \quad n = \sin \theta
\]

(5)

Therefore, the effective elastic properties of the PMNC surrounding the carbon fiber with respect to the principle material coordinate axes of the SFFRC vary over an annular cross section of the PMNC phase of the RVE of the SCFF, as shown in Figs. 4 and 5(a). However, without loss of generality, it may be considered that the volume average of these effective elastic properties \([C_{nc}^{\text{eff}}]\) over the volume of the PMNC can be treated as the constant effective elastic properties \([C_{c}^{\text{eff}}]\) of the PMNC material surrounding the carbon fiber with respect to the principle coordinate axes \((1–2–3)\) of the SFFRC and are given in Ref. [58] as follows:

\[
[C_{c}^{\text{eff}}] = \frac{1}{\pi (b^2 - a^2)^2} \int_a^b [C_{nc}^{\text{eff}}] r \, dr \, d\theta
\]

(6)

It may be mentioned here that if the CNT volume fraction is homogenized in the annular portion of the RVE of the PMNC, the homogenized effective elastic properties of the PMNC will not be radially dependent (see Appendix of Ref. [51] for details). Recently, Chatzigeorgiou et al. [26] determined the elastic properties of the CNT-reinforced nanocomposite with and without considering the radial dependency of the elastic properties of the nanocomposite. Subsequently, they introduced the homogenized nanocomposite in the actual “fuzzy fiber” composite. In their study, a good agreement between the two sets of results (with and without considering the radial dependency of the elastic properties of the nanocomposite) predicting the effective elastic properties of the fuzzy fiber composites has been found.

4 Shear Lag Model Formulation

Lamina of the SFFRC can be viewed to be composed of either the RVEs-A or the RVEs-B, arranged in a periodic manner in the polymer matrix, as illustrated in Fig. 2(b). Figure 5(a) illustrates the cross sections of the cylindrical RVE-A based on which analytical three-phase shear lag model is derived herein. It may be noted that many pioneering researchers [31–33,35–38,59] considered such cylindrical RVE in their two-phase and three-phase shear lag studies. The SFFRC and its constituent phases are being studied here are made of aligned reinforcements. Therefore, a three-phase shear lag model has been developed in this section considering the orthotropy of the SFFRC and its constituent phases. In our model, the application of the shear stress \((\tau)\) along the length of the RVE at \(r = R\) [40–46,48] accounts for the staggering of the adjacent RVEs-A, whereas the consideration of the
radial load \( q \) on the RVE accounts for the lateral extensional interaction between the adjacent RVEs. Such radial/residual stresses may arise due to the manufacturing processes, such as injection molding of the short fiber composites.

The cylindrical coordinate system \((\rho, \theta, z)\) is considered in such a way that the axis of the RVE coincides with the \( x \)-axis, while CNTs are aligned along the \( r \)-direction. The model is derived by dividing the RVE into three zones. The portion of the RVE in the zone \( -L_1 \leq x \leq -L_4 \) consists of three concentric cylindrical phases; namely, the carbon fiber, the PMNC, and the polymer matrix. The RVE of the SFFRC has the radius \( R \) and the length \( 2L \). The radius and the length of the carbon fiber are denoted by \( a \) and \( 2L_1 \), respectively. The inner and outer radii of the PMNC phase are \( a \) and \( b \), respectively. The respective portions of the RVE in the zones \( -L \leq x \leq -L_4 \) and \( L_1 \leq x \leq L \) are considered to be composed of an imaginary fiber, an imaginary PMNC, and the polymer matrix phase. The radius of an imaginary fiber is also denoted by \( a \), while the inner and outer radii of an imaginary PMNC phase are also represented by \( a \) and \( b \), respectively. Thus, the shear lag model derived for the zone \( -L_4 \leq x \leq -L_2 \) can be applied to derive the shear lag models for the zones \( -L \leq x \leq -L_4 \) and \( L_1 \leq x \leq L \). In what follows, the shear lag model for the zone \( -L_4 \leq x \leq -L_2 \) is first derived.

### 4.1 Governing Equations of the Stress Transfer Analysis for the Middle Portion \((- L_1 \leq x \leq L_1 \)) of the RVE—A

The governing equilibrium equations for an axisymmetric RVE problem in terms of the cylindrical coordinates \((r, \theta, z)\) are given in Ref. [60]

\[
\frac{\partial \sigma_r^l}{\partial r} + \frac{\partial \sigma_r^l}{\partial r} + \frac{\sigma_r^l - \sigma_z^l}{r} = 0; \quad k = f, c, \quad \text{and} \quad m \tag{7a}
\]

and

\[
\frac{\partial \sigma_r^l}{\partial r} + \frac{1}{r} \frac{\partial (r \sigma_r^l)}{\partial r} = 0; \quad k = f, c, \quad \text{and} \quad m \tag{7b}
\]

while the relevant constitutive relations are

\[
\sigma_r^f = C_{11} \varepsilon_r^f + C_{12} \varepsilon_\theta^f + C_{13} \varepsilon_z^f; \quad \sigma_r^c = C_{11} \varepsilon_r^c + C_{21} \varepsilon_\theta^c + C_{31} \varepsilon_z^c
\]

and

\[
\sigma_r^m = C_{66} \varepsilon_r^m; \quad k = f, c, \quad \text{and} \quad m \tag{8}
\]

In Eqs. (7) and (8), the respective superscripts \( f, c, \) and \( m \) denote the carbon fiber, the PMNC, and the polymer matrix. For the \( k \)th constituent phase, \( \sigma_{r,k}^f \) and \( \sigma_{r,k}^c \) represent the normal stresses in the \( x \) and \( r \) directions, respectively; \( \varepsilon_{r,k}^f \) and \( \varepsilon_{r,k}^c \) are the normal strains along the \( x, \theta \), and \( r \), directions, respectively; \( \varepsilon_{r,k}^m \) is the transverse shear stress, \( \varepsilon_{r,k}^m \) is the transverse shear strain, and \( C_{ik}^k \) are the elastic constants. The strain–displacement relations for an axisymmetric problem relevant to this RVE are

\[
\varepsilon_{r,k}^f = \frac{\partial u_k^f}{\partial r}, \quad \varepsilon_{r,k}^c = \frac{\partial u_k^c}{\partial r}, \quad \text{and} \quad \varepsilon_{r,k}^m = \frac{\partial u_k^m}{\partial r} + \frac{\partial u_k^m}{\partial x} \tag{9}
\]

in which \( u_k^f \) and \( u_k^c \) represent the axial and radial displacements at any point of the \( k \)th phase along the \( x \) and \( r \) directions, respectively. The traction boundary conditions are given by

\[
\sigma_{r,k}^m |_{x=R} = q \quad \text{and} \quad \sigma_{r,k}^m |_{x=-R} = \tau \tag{10}
\]

and the interfacial traction continuity conditions are given by

\[
\frac{\partial \sigma_{r,k}^f}{\partial r} = \frac{\partial \sigma_{r,k}^c}{\partial r}; \quad c \quad \text{and} \quad m \tag{14}
\]

Thus, using the equilibrium equation given by Eq. (7b), the transverse shear stresses in the PMNC phase and the polymer matrix phase can be expressed in terms of the interfacial shear stresses \( \tau_1 \) and \( \tau_2 \), respectively, as follows:

\[
\varepsilon_{r,k}^c = \frac{a}{r} \tau_1 + \frac{1}{2r} \left( a^2 - r^2 \right) \frac{\partial \sigma_{r,k}^c}{\partial x} \tag{15}
\]

and

\[
\varepsilon_{r,k}^m = \frac{a^2}{2r^2} \tau_2 + \frac{R^2 - r^2}{R^2} \tau_2 - \frac{R}{r} \tau \tag{16}
\]

Also, since the RVE is an axisymmetric problem, it is usually assumed [47,59] that the gradient of the radial displacements with respect to the \( x \)-direction is negligible and so, from the constitutive relation given by Eq. (8) and the strain–displacement relations given by Eq. (9) between \( \sigma_{r,k}^c \) and \( \varepsilon_{r,k}^c \), one can write

\[
\frac{\partial u_k^c}{\partial r} \approx \frac{a}{C_{66}^k} \varepsilon_{r,k}^c; \quad c \quad \text{and} \quad m \tag{17}
\]

Solving Eq. (17) and satisfying the continuity conditions at \( r = a \) and \( r = b \), respectively, the axial displacements of the PMNC

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The expressions of the coefficients $b_i$ can be derived as follows:

$$u^c = u_0^c + A_1 \tau_1 + A_2 \tau_2$$  \hspace{1cm}  \text{(18)}$$
$$u^m = u_0^m + A_3 \tau_1 + A_4 \tau_2 + \frac{R}{C_{66}} \ln \left( \frac{C}{R} \right)$$  \hspace{1cm}  \text{(19)}$$

and

$$u_y^c = u_y^c \big|_{r=a}$$  \hspace{1cm}  \text{(20)}$$

in which $A_i$ ($i = 1, 2, 3$ and 4) are the constants of the displacement fields of the PMNC and the polymer matrix, and are explicitly shown in the Appendix.

The radial displacements in the three constituent phases can be assumed as [61]

$$w^c = C_1 r, \quad w^m = C_2 r + \frac{B_s}{r} \quad \text{and} \quad w^m = C_3 r$$  \hspace{1cm}  \text{(21)}$$

where $C_1, A_3, B_s, C_2, \text{and} C_3$ are the unknown constants. Invoking the continuity conditions for the radial displacement at the interface $r = a$ and $b$, the radial displacement in the PMNC phase can be augmented as follows:

$$w^c = \frac{a^2}{b^2 - a^2} \left( \frac{b^2}{r} - 1 \right) C_1 - \frac{b^2}{r^2} \left( \frac{a^2}{r} - 1 \right) C_2 - \frac{1}{r^2} \left( \frac{a^2}{r} - 1 \right) C_3$$  \hspace{1cm}  \text{(22)}$$

Substituting Eqs. (18), (19), (21), and (22) into Eq. (9) and subsequently, employing the constitutive relations (8), the expressions for the normal stresses can be written in terms of the unknown constants $C_1, C_2$, and $C_3$ as follows:

$$\sigma_i^c = C_{11} \sigma_i^c + 2C_{12} C_1$$  \hspace{1cm}  \text{(23)}$$

The expressions of the coefficients $B_i$ are presented in the Appendix. Solving Eq. (29), the solutions of the constants $C_1, C_2$, and $C_3$ can be expressed as

$$C_i = b_0 \sigma_i^c + b_2 \tau_i^c + b_3 \tau_i^m + b_4 q + b_5 \tau; \quad i = 1, 2, 3$$  \hspace{1cm}  \text{(30)}$$

The expressions of the coefficients $h_1, h_2, h_3, b_4$, and $b_5$ are evident from Eq. (30), and are not shown here for the sake of clarity. Now, making use of Eqs. (25), (27), and (30) in the last two equations of Eq. (12), respectively, the average axial stresses in the PMNC phase and the polymer matrix phase are written as follows:

$$\sigma_i^c = \frac{C_{11}}{C_{11}} \sigma_i^c + \left[ C_{11} + C_{13} - \frac{2(C_{13})^2}{C_{13}} \right] C_1$$  \hspace{1cm}  \text{(24)}$$
$$\sigma_i^m = \frac{C_{11}}{C_{11}} \sigma_i^m + \left[ 2C_{13} \frac{a^2}{b^2 - a^2} + \frac{b^2}{b^2 - a^2} \left( b^2 - a^2 \right) \right] C_1$$
$$+ \frac{2C_{12}}{b^2 - a^2} C_3 + C_{11} A_1 \tau_1^c + C_{11} A_2 \tau_2^c$$  \hspace{1cm}  \text{(25)}$$

$$\sigma_i^c = \frac{C_{11}}{C_{11}} \sigma_i^c + \left[ \frac{C_{13} a^2}{b^2 - a^2} + \frac{C_{11} b^2}{b^2 - a^2} \right] C_1$$
$$+ \frac{C_{13}}{b^2 - a^2} \left( a^2 - 1 \right) C_2$$
$$+ \frac{C_{23}}{b^2 - a^2} \left( a^2 - 1 \right) C_3$$
$$+ A_1 A_1 \tau_1^c + C_{11} A_2 \tau_2^c$$  \hspace{1cm}  \text{(26)}$$

where the prime notations denote the differentiation with respect to the axial coordinate ($\xi$). Invoking the continuity conditions $\sigma_i^c \big|_{r=a} - \sigma_i^m \big|_{r=a}$ and $\sigma_i^c \big|_{r=b} = \sigma_i^m \big|_{r=b}$, and satisfying the boundary condition $\sigma_i^m \big|_{r=b} = q$, the following equations for solving $C_1, C_2$, and $C_3$ are obtained:

$$\sigma_i^c = \frac{C_{11}}{C_{11}} \sigma_i^c + \left[ C_{11} - C_{13} \right] C_1$$
$$+ C_{11} A_1 \tau_1^c + \frac{R}{C_{11}} \ln \left( \frac{C}{R} \right) C_{11}$$  \hspace{1cm}  \text{(27)}$$

$$\sigma_i^m = \frac{C_{11}}{C_{11}} \sigma_i^m + \left[ C_{11} - C_{13} \right] C_1$$
$$+ C_{11} A_1 \tau_1^c + \frac{R}{C_{11}} \ln \left( \frac{1}{b} \right) C_{11}$$  \hspace{1cm}  \text{(28)}$$

The constants $A_i$ ($i = 5, 6, 7, \ldots, 24$) appeared in the four equations are presented in the Appendix. Now, satisfying the equilibrium of force along the axial ($\xi$) direction at any transverse cross section of the RVE-A, the following equation is obtained:

$$\pi R^2 \sigma = \pi a^2 \sigma_i + \pi \left( b^2 - a^2 \right) \sigma_i^c + \pi \left( R^2 - b^2 \right) \sigma_i^m$$  \hspace{1cm}  \text{(33)}$$
Differentiating the first and last equations of Eq. (13) with respect to $x$, we have

$$
\tau_1' = \frac{a}{2} \sigma_x'' \quad \text{and} \quad \tau_2' = \frac{R^2 - h^2}{2b} \sigma_m''
$$

(34)

Use of Eqs. (31)–(34), yields

$$
A_{23} \sigma_1'' + A_{23} \sigma_2'' + A_{23} q + A_{23} \tau - R^2 \tau = 0
$$

(35)

Deriving the expression for $\tau_2'$ from Eq. (31) and substituting the same into Eq. (32), and then using Eq. (34), the following result for the average axial stress in the carbon fiber coated with the radii-

$$
\sigma_2' = \left(20 - \frac{1}{2} A_{11} \frac{A}{16} \right) \sigma_x + \left( \frac{a}{2} \right) \left( \frac{A_{22} A_{13}}{A_{16}} - A_{22} \right) \sigma_x''
$$

(36)

Differentiating Eqs. (33) and (36) twice with respect to $x$ and using the resulting two equations in Eq. (35), the governing equation for the average axial stress in the carbon fiber coated with the radially grown aligned CNTs is obtained as follows:

$$
\sigma_x'(4) + A_{31} \sigma_x'' + A_{12} \sigma_1 + A_{33} q + A_{35} \tau = 0
$$

(37)

The constants $A_i (i = 25, 26, 27, ..., 35)$ appeared in the above three equations are presented in the Appendix. Solution of Eq. (37) is given by

$$
\sigma_x(4) + A_{31} \sigma_x'' + A_{12} \sigma_1 + A_{33} q + A_{35} \tau = 0
$$

(38)

where

$$
\alpha = \sqrt{1/2 \left( -A_{31} + \sqrt{\left( A_{31}\right)^2 - 4A_{32}} \right)} \quad \text{and} \quad \beta = \sqrt{1/2 \left( -A_{31} - \sqrt{\left( A_{31}\right)^2 - 4A_{32}} \right)}
$$

(39)

Substitution of Eq. (39) into the first equation of Eq. (13) yields the expression for the carbon fiber/PMNC interfacial shear stress as follows:

$$
\tau_1 = \frac{a}{2} \left[ A_{36} x \sinh(xz) + A_{37} \cosh(xz) + A_{38} \sinh(\beta x) + A_{39} \cosh(\beta x) \right]
$$

(40)

4.2 Governing Equations of the Stress Transfer Analysis for the Two-End Portions ($-L \leq x \leq -L_i$ and $L_i \leq x \leq L$) of the RVE-A. The two-end portions of the RVE-A in the zones $-L \leq x \leq -L_i$ and $L_i \leq x \leq L$ are considered to comprise imaginary carbon fiber (pf) made of the polymer material, an imaginary PMNC (pc) made of the polymer material and the polymer matrix. The radius of the imaginary fiber is also denoted by $a$, while the inner and outer radii of the imaginary PMNC phase are also represented by $a$ and $b$, respectively. Thus, the solutions derived for the middle portion of the RVE-A ($-L_i \leq x \leq L_i$) can be applied to derive the solutions for the zones $-L \leq x \leq -L_i$ and $L_i \leq x \leq L$ by substituting $C^{pf}_i = C^{pc}_i = C_i$. The traction boundary conditions and the interfacial continuity conditions are given by

$$
\sigma_x'(x = \pm L) = \sigma_x(\pm L) \quad \text{and} \quad \tau_1(x = \pm L) = 0
$$

(41a)

Utilizing the end conditions given by Eq. (41a) in Eq. (45), the constants $A_{36}^{pf}$, $A_{37}^{pf}$, $A_{38}^{pf}$ and $A_{39}^{pf}$ can be explicitly written as follows:

$$
A_{36}^{pf} = 0
$$

(46)

$$
A_{37}^{pf} = -\beta \sin h(\beta^{pf} L)
$$

(47)

$$
A_{38}^{pf} = 0
$$

(48)

$$
A_{39}^{pf} = -\beta \sin h(\beta^{pf} L)
$$

(49)
Substituting Eqs. (46)–(49) in Eq. (45), the final solution for $\sigma_{s}^{ef}$ is obtained as follows:

$$
\sigma_{s}^{ef} = \left[ \frac{\beta d_{s}^{2} \sin (\beta d_{s} L) \cos (\beta d_{s} x) - \beta d_{s}^{2} \sin (\beta d_{s} L) \cos (\beta d_{s} x)}{\beta d_{s}^{2} \sin (\beta d_{s} L) \cos (\beta d_{s} x) - \beta d_{s}^{2} \sin (\beta d_{s} L) \cos (\beta d_{s} L)} \right] \\
\times \left\{ \frac{A_{32}^{g} \sigma + A_{33}^{g} q + A_{34}^{g} \tau}{A_{32}^{g} \sigma + A_{33}^{g} q + A_{34}^{g} \tau} + \frac{A_{32}^{g} \sigma - A_{33}^{g} q - A_{34}^{g} \tau}{A_{32}^{g} \sigma + A_{33}^{g} q + A_{34}^{g} \tau} \right\}
$$

(50)

Similarly, utilizing Eq. (45) and the end conditions given by Eq. (41) in Eq. (38), the constants $A_{36}$, $A_{37}$, $A_{38}$, and $A_{39}$ are evaluated to determine $\sigma_{s}^{ef}$.

5 Results and Discussion

In this section, the predictions by the MT model are first compared with those of the existing experimental and numerical results of the SCFF and the CNT-reinforced composite to verify the validity of the SFFRC modeling. Subsequently, the results from the analytical shear lag model are compared with the FE shear lag simulations and are used to analyze the stress transfer characteristics of the SFFRC. Zig-zag CNTs, carbon fiber, and polyimide matrix are used for evaluating the numerical results. Their material properties, available in Refs. [8,62,63], are listed in Table 1. The effective elastic properties and the thickness of the hollow circular cylindrical continuum representing the interphase between a CNT and the polyimide matrix are also presented in Table 1. Unless otherwise mentioned, the following geometrical parameters of the RVE of the SFFRC are adopted to compute the results in the present study:

- volume fraction of the carbon fiber in the SFFRC ($v_{f}$) = 0.4
- volume fraction of the SCFF in the RVE-A ($v_{SCFF}$) = 0.7513
- radius of the carbon fiber ($a$) = 5 $\mu$m
- radius of the SCFF ($b$) = 6.5258 $\mu$m

Table 1 Material properties of the constituent phases of the SFFRC

<table>
<thead>
<tr>
<th>Material</th>
<th>$C_{11}$ (GPa)</th>
<th>$C_{12}$ (GPa)</th>
<th>$C_{13}$ (GPa)</th>
<th>$C_{33}$ (GPa)</th>
<th>$C_{44}$ (GPa)</th>
<th>$C_{66}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 0) CNT</td>
<td>709.9</td>
<td>172.4</td>
<td>240</td>
<td>240</td>
<td>240</td>
<td>1513.1</td>
</tr>
<tr>
<td>Interphase</td>
<td>29.6</td>
<td>15.2</td>
<td>15.2</td>
<td>15.2</td>
<td>29.6</td>
<td>7.2</td>
</tr>
<tr>
<td>Carbon fiber</td>
<td>236.4</td>
<td>10.6</td>
<td>10.6</td>
<td>10.7</td>
<td>24.8</td>
<td>7.3</td>
</tr>
<tr>
<td>Polyimide</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>9</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Table 7 Transverse and longitudinal cross sections of the NRLC (from Ref. [14])

Fig. 7 Transverse cross section of the hexagonal packing array comprising SCFFs and the polymer matrix (from Ref. [27])

Fig. 6 Transverse cross section of the hexagonal packing array comprising SCFFs and the polymer matrix (from Ref. [27])

The determination of the CNT volume fraction ($V_{CNT}$) in the SFFRC is an important issue. It is obvious that the constructional feature of the SFFRC imposes a constraint on the maximum value of $V_{CNT}$. Since in the PMNC material, polymer molecules fill the gap between CNTs and the formation of the interphase is also considered between a CNT and the polymer, the surface-to-surface distance between two adjacent CNTs at their roots is considered as 1.7 nm [25]. In the SFFRC, hexagonal packing array of the SCFFs is considered as shown in Fig. 6 for evaluating the numerical results while they are not touching each other. Based on the carbon fiber volume fraction ($V_{f}$) in the SFFRC, the maximum values of the CNT volume fractions in the SFFRC ($V_{CNT}$) and the PMNC ($V_{n}$) can be determined as follows [27]:

$$
V_{CNT} = \frac{V_{CNT}}{V_{PMNC}} = \frac{\pi d_{s}^{2}}{2(a + 1.7)} \left( b \frac{2}{a - 1} - 1 \right) v_{f}
$$

(51a)

$$
V_{n} = \frac{V_{CNT}}{V_{PMNC}} = \frac{(V_{CNT})_{\text{max}}}{b^{2} \frac{v_{f}}{a - 1}}
$$

(51b)

5.1 Comparisons With Experimental and Numerical Results. Kulkarni et al. [14] investigated experimentally and numerically the elastic response of the nanoreinforced laminated composite (NRLC). The NRLC is made of the CNT-reinforced polymer nanocomposite and the carbon fiber. The cross sections of such NRLC are shown schematically in Fig. 7. The geometry of the NRLC as depicted in Fig. 7 is similar to that of the SCFF presented in Fig. 5(a). Thus, to validate the modeling of the SCFF in the present study, the comparisons have been made between the results predicted by Kulkarni et al. [14] for the NRLC with the results predicted by the MT model for the SCFF. It may be observed from Table 2 that the predicted value of the transverse
Young’s modulus ($E_y$) of the SCFF computed by the MT model match closely with the experimental value predicted by Kulkarni et al. [14]. The experimental value of $E_y$ is lower than the theoretical prediction, and this may be attributed to the fact that CNTs are not perfectly radially grown and straight, and hence the radial stiffening of the NRLC decreases [14]. It may also be noted that the value of $E_y$ predicted by the MT model utilized herein is much closer to the experimental value than that of the numerical value predicted by Kulkarni et al. [14]. This is attributed to the fact that the appropriate transformation and homogenization procedures given by Eqs. (4) and (6) have been employed in the present study, whereas Kulkarni et al. [14] did not consider such transformation and homogenization procedures in their numerical modeling. These comparisons are significant since the prediction of the transverse Young’s modulus of the SCFF provides critical check for the validity of the MT model. Thus, it can be inferred from the comparisons shown in Table 2 that the MT model can be reliably applied to predict the elastic properties of the PMNC.

Although the applicability of the MT model for the modeling of the SCFF has been confirmed, the work was further extended to validate the MT model against that of the FE model predictions of the elastic properties of the unwound PMNC lamina. In order to verify the validity of the MT model, the CNTs and the matrix material of the nanocomposite studied by Liu and Chen [9] are considered for the constituent phases of the unwound PMNC material. The engineering constants of this unwound PMNC material computed by the MT model have been compared with those of the same predicted by Liu and Chen [9]. Table 3 illustrates this comparison and it may be observed that the two sets of results are in excellent agreement and thus validating the MT model employed in this study.

![Table 2 Comparisons of the effective engineering constants of the NRLC with those of the SCFF containing straight CNTs](attachment:table2.png)

<table>
<thead>
<tr>
<th>NRLC (2% CNT and 41% IM7 carbon fiber) [14]</th>
<th>Numerical</th>
<th>Experimental</th>
<th>MT model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$ (GPa)</td>
<td>13.93</td>
<td>10.02</td>
<td>11.91</td>
</tr>
<tr>
<td>$\nu_{xy}$</td>
<td>0.34</td>
<td>—</td>
<td>0.38</td>
</tr>
<tr>
<td>$\nu_{yx}$</td>
<td>0.16</td>
<td>—</td>
<td>0.18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3 Comparison of the engineering constants of the unwound PMNC material</th>
<th>$E'/E_y$</th>
<th>$E_1/E_y$</th>
<th>$E_2/E_y$</th>
<th>$\nu_{12}, \nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.1948</td>
<td>1.1948</td>
<td>1.1737</td>
<td>1.0666</td>
</tr>
<tr>
<td>10</td>
<td>1.4384</td>
<td>1.4384</td>
<td>1.3336</td>
<td>1.0912</td>
</tr>
</tbody>
</table>

Note: $E' = 1000$ GPa, $\nu^p = 0.3$, $\nu^s = 0.3$, and CNT volume fraction, $v_0 = 0.04871$ [9]; where $E'$ and $E_y$ are the Young’s moduli of the CNT and the polymer matrix, respectively; $E_1$ and $E_2$ are the axial and the transverse Young’s moduli of the unwound PMNC, respectively; $\nu^p$ and $\nu^s$ are the Poisson’s ratios of the CNT and the polymer matrix, respectively; $\nu_{12}$ and $\nu_{23}$ are the axial and the transverse Poisson’s ratios of the unwound PMNC, respectively.

$$\{\bar{\sigma}^k\} = \frac{1}{V}\int\{\sigma^k\}dV; \quad k = f, c, \text{ and } m$$  \hspace{1cm} (52)

where $\bar{\sigma}^k$ represents the volume of the $k$th phase of the RVE-B and the field variable with an overbar represents the average of the field variable. For analyzing the stress transfer characteristics, the following nondimensional parameters are used:

$$\sigma^* = \frac{\bar{\sigma}^f}{\sigma} \quad \text{and} \quad \tau^* = \frac{\bar{\tau}^c}{\sigma}$$  \hspace{1cm} (53)

It should be noted that the FE shear lag model accounts for the interaction among the neighboring phases and provides the overall load transfer characteristics of the SFFRC. Tension–shear model with the regular staggering has been introduced by Gao and coworkers [41–46] to interpret many underlying mechanisms in biocomposites. Their model yielded simplified expressions for the effective stiffness, fracture toughness, buckling strength, shearing stresses, and these predictions are also found to be compatible with the FE simulations. According to Gao and coworkers, the axial tensile stress ($\sigma$) imposed on the RVE is related to the shear stress ($\tau$) along the RVE-A as shown in Fig. 5(a) is given by

$$\sigma = \frac{L}{2R}v_{SCFF}\tau$$  \hspace{1cm} (54)

in which $v_{SCFF}$ is volume fraction of the SCFF in the RVE-A. The comparisons of the average axial stress in the carbon fiber ($\sigma^*$) and the interfacial shear stress between the carbon fiber and the PMNC ($\tau^*$) computed by the analytical shear lag model and the FE shear lag model are presented in Figs. 9 and 10, respectively. It should be noted that because of symmetry, distributions of the stresses in the zone of the carbon fiber reinforcement are plotted for one half of the carbon fiber only. It may be importantly observed from Figs. 9 and 10 that if the staggering effect of the adjacent RVEs-A is neglected (that is, $\tau = 0$), then the analytical shear lag model overpredict values of $\sigma^*$ and $\tau^*$ compared with those of the analytical and the FE shear lag models predictions considering the staggering effect of the adjacent RVEs-A (that is, $\tau \neq 0$). Hence, the existence of the non-negligible shear tractions along the length of the RVE of the SFFRC cannot be ignored; they play a crucial role in the stress transfer characteristics of the SFFRC. It may also be observed that the good agreement between the two sets of results has been obtained and thus verifying the reliability of the present analytical shear lag model incorporating the staggering effect of the adjacent RVEs-A.

5.2 Analytical Shear Lag Model Results. In this section, parametric results of a shear lag analysis of the hybrid hierarchical SFFRC have been presented to investigate the effects of variations of the radial load ($q$) on the RVE and the spacings between the adjacent SCFFs. The consideration of the effect of the radial load on the RVE accounts for the lateral extensional interaction between
the adjacent SCFFs, which may arise due to the manufacturing processes. Therefore, the application of the radial load on the RVE is considered in the range from 0 to 0.5 \( r \). The variations of the values of \( r/C_3 \) and \( s/C_3 \) are presented in Figs. 11 and 12 for the different values of \( q \). It may be observed from these figures that the maximum values of \( r/C_3 \) and \( s/C_3 \) are significantly decreased with the increase in the magnitude of the radial load. When compared with the results without application of the radial load, it is observed that almost 25% and 65% reductions in the maximum values of \( r/C_3 \) and \( s/C_3 \) occur if the value of \( q = 0.5 \).

So far, in this work, the stress transfer characteristics of the SFFRC have been studied by considering the values of the geometrical parameters \( R/b = L/L_f \) as 1.1. Here, the geometrical parameters \( R/b \) and \( L/L_f \) represent the spacings between the adjacent SCFFs along their radial and axial directions, respectively, over the volume of the SFFRC lamina. Practically, the gaps between the adjacent SCFFs for a particular value of \( v_f \) would be an important study. For this the four discrete values of \( R/b = L/L_f \) are considered as 1.05, 1.1, 1.15 and 1.2. Keeping the diameter of the carbon fiber constant as 2 \( a = 10 \) \( l_m \), the effective elastic coefficients of the PMNC corresponding to these geometrical parameters have been determined and are listed in Table 4. The variations of the values of \( r/C_3 \) and \( s/C_3 \) are presented in Figs. 13 and 14 for different values of \( R/b = L/L_f \) when the radial load is absent (\( q = 0 \)). The maximum value of \( r/C_3 \) decreases with the increase in the values of \( R/b \) and \( L/L_f \) as illustrated in Fig. 13. This is attributed to the fact that the effective transverse elastic coefficients \( C_{32}, C_{33}, C_{34}, \) and \( C_{44} \) of the PMNC are improved and the staggering effect of the adjacent RVEs is increased with the increase in the values of \( R/b \) and \( L/L_f \) ratios which eventually enhance the load carrying capacity of the PMNC in the radial direction. The maximum value of \( r/C_3 \) is found

<table>
<thead>
<tr>
<th>( R/b = L/L_f )</th>
<th>( V_{CNT} )</th>
<th>( v_\text{n} )</th>
<th>( C_{11} ) (GPa)</th>
<th>( C_{12} ) (GPa)</th>
<th>( C_{22} ) (GPa)</th>
<th>( C_{33} ) (GPa)</th>
<th>( C_{44} ) (GPa)</th>
<th>( C_{66} ) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05</td>
<td>0.0248</td>
<td>0.0648</td>
<td>11.388</td>
<td>7.167</td>
<td>18.136</td>
<td>45.314</td>
<td>13.589</td>
<td>2.528</td>
</tr>
<tr>
<td>1.1</td>
<td>0.019</td>
<td>0.0674</td>
<td>11.507</td>
<td>7.225</td>
<td>18.639</td>
<td>46.816</td>
<td>14.089</td>
<td>2.578</td>
</tr>
<tr>
<td>1.15</td>
<td>0.0137</td>
<td>0.07</td>
<td>11.624</td>
<td>7.281</td>
<td>19.125</td>
<td>48.266</td>
<td>14.57</td>
<td>2.626</td>
</tr>
<tr>
<td>1.2</td>
<td>0.009</td>
<td>0.0724</td>
<td>11.738</td>
<td>7.337</td>
<td>19.594</td>
<td>49.665</td>
<td>15.036</td>
<td>2.673</td>
</tr>
</tbody>
</table>
to be marginally affected by the variation of the values of \( R/b \) and \( L/L_f \) as shown in Fig. 14. Although not shown here, the maximum values of \( r/C_3 \) and \( s/C_3 \) are significantly decreased with the application of the radial load on the RVE of the SFFRC.

In the previous sets of results, the stress transfer characteristics of the SFFRC have been investigated by applying the tensile radial load on the RVE for the different values of \( R/b \) and \( L/L_f \).

However, variation of the radial load from compressive to tensile may affect the stress transfer characteristics of the SFFRC. Once again, the four discrete values of \( R/b \) and \( L/L_f \) are considered as 1.05, 1.1, 1.15, and 1.2 to investigate the effect of the variation of the radial load. The variations of the values of \( r/C_3 \) and \( s/C_3 \) are presented in Figs. 15 and 16 for different values of \( q \), \( R/b \), and \( L/L_f \). Two specific locations over the length of the carbon fiber where the maximum stresses \( (r/C_3 \) and \( s/C_3) \) occur are considered for computing the results as presented in Figs. 15 and 16. It may be observed from these
figures that if the applied radial load is compressive, then the maximum values of $\sigma^r$ and $\tau^r$ are higher than those without the application of the radial load ($q = 0$) and vice versa. It is important to note from Figs. 13–16 that if the values of $R/b = L/L_4 = 1.2$ and the corresponding CNT volume fraction in the SFFRC is 0.9%, then the maximum values of $\sigma^r$ and $\tau^r$ are significantly decreased irrespective of the magnitude of the applied radial load on the RVE signifying the fact that the value of the optimum spacing between the adjacent SCFFs interlaced in the polymer matrix may be considered as $R/b = L/L_4 = 1.2$. It may be noted that the stress transfer characteristics of the SFFRC may further be improved by optimizing the structural and material parameters. However, these studies are considered to be outside the scope of the present effort.

6 Conclusions

In this article, we investigated the stress transfer characteristics of a novel hybrid hierarchical SFFRC incorporating the staggering effect of the adjacent RVEs. The distinctive feature of the construction of the SFFRC is that the fuzzy fibers made of the short carbon fiber reinforcements coated with the radially aligned CNTs on their circumferential surfaces are regularly staggered over the volume of the SFFRC lamina. The following main inferences are drawn from our study:

(1) The existence of the non-negligible shear tractions along the length of the RVE of the SFFRC cannot be neglected; they play a crucial role in the stress transfer characteristics of the hierarchical periodic nanocomposites.

(2) For the higher values of the applied tensile radial loads, the maximum values of the axial stress transferred to the carbon fiber and the interfacial shear stress along its length are significantly decreased.

(3) The load transfer characteristics of the SFFRC are significantly improved with the increase in the spacings between the adjacent SCFFs in the polymer matrix irrespective of the magnitude of the radial load.

(4) The three-phase shear lag model developed in this study is capable of analyzing the mechanisms of load transfer between the orthotropic constituent phases of any advanced composite subjected to three-dimensional loads.

Acknowledgment

This work is funded by the Natural Sciences and Engineering Research Council (NSERC) of Canada and the Discovery Accelerated Supplement by NSERC. The authors wish to thank the reviewers for their helpful comments.

Nomenclature

Notations

- $a = \text{radius of the carbon fiber (m)}$
- $[A_{n}], [A_{m}] = \text{matrices of the strain concentration factors}$
- $b = \text{radius of the SCFF (m)}$
- $[C^k] = \text{elastic coefficient matrix of the } k\text{th phase (GPa)}$
- $[C^{nc}] = \text{elastic coefficient matrix of the unwound PMNC (GPa)}$
- $C_{ij}^k = \text{elastic coefficients of the } k\text{th phase (GPa)}$
- $d_{0} = \text{diameter of the CNT (m)}$
- $E_a, E_p = \text{Young's moduli of the CNT and the polymer, respectively (GPa)}$
- $E_1, E_2 = \text{axial and transverse Young's moduli of the CNT-reinforced composite, respectively (GPa)}$
- $[I] = \text{fourth order identity matrix}$
- $L = \text{half length of the RVE of the SFFRC (m)}$
- $L_4 = \text{half length of the short carbon fiber (m)}$
- $q = \text{radial normal stress applied on the RVE-A of the SFFRC in the } r\text{-direction (GPa)}$
- $R = \text{radius of the RVE-A/width of the RVE-B of the SFFRC (m)}$
- $[S_{n}], [S_{m}] = \text{Eshelby tensors for the CNT and the interphase, respectively}$
- $[T] = \text{transformation matrix}$
- $u^k, w^k = \text{axial and radial displacements at any point in the } k\text{th phase along the } x\text{- and } r\text{-directions, respectively (m)}$
- $V_{CNT} = \text{CNT volume fraction in the SFFRC}$
- $V_1 = \text{carbon fiber volume fraction in the SFFRC}$
- $V_m = \text{polymer volume fraction in the PMNC}$
- $V_n = \text{CNT volume fraction in the PMNC/CNT-reinforced nanocomposite}$
- $V^k = \text{volume of the } k\text{th phase (m$^3$)}$

Greek Symbols

- $\epsilon^k_x, \epsilon^k_{\theta}, \epsilon^k_r = \text{normal strains along the } x, \theta, \text{ and } r\text{ directions, respectively, in the } k\text{th phase}$
- $\epsilon^k_{\theta} = \text{transverse shear strain in the } k\text{th phase}$
- $\theta = \text{angle between the radial axis (3}-\text{axis) along which the CNT is grown and the 2–3 axis in the 2–3 plane}$
- $\nu_{12}, \nu_{23} = \text{axial and transverse Poisson’s ratios of the CNT-reinforced composite, respectively}$
- $\nu^p_x, \nu^p_{xy} = \text{Poisson’s ratios of the CNT and the polymer, respectively}$
- $\sigma = \text{axial normal stress applied on the RVE of the SFFRC in the } x\text{-direction (GPa)}$
- $\sigma^r = \text{non-dimensional axial stress along the length of the carbon fiber}$
- $\sigma^k = \text{stress vector of the } k\text{th phase (GPa)}$
- $\sigma^k_{ij} = \text{average stress vector of the } k\text{th phase (GPa)}$
- $\sigma^r_{ij} = \text{normal stresses along the } x\text{- and } r\text{-directions, respectively, in the } k\text{th phase (GPa)}$
- $\sigma^k = \text{transverse shear strain in the } k\text{th phase (GPa)}$
- $\sigma^r_{ij} = \text{average axial stress in the } k\text{th phase (GPa)}$
- $\sigma^p_{ij} = \text{average axial stress in an imaginary carbon fiber made of the polymer material (GPa)}$
- $\sigma^{pm}_{ij} = \text{average axial stress in a CNT made of the polymer material (GPa)}$
- $\tau = \text{shear stress along the length of the RVE-A of the SFFRC (GPa)}$
- $\tau_1 = \text{transverse shear stress at the interface between the carbon fiber and the PMNC (GPa)}$
- $\tau_1^r = \text{nondimensional transverse shear stress at the interface between the carbon fiber and the PMNC}$
- $\tau_2 = \text{transverse shear stress at the interface between the PMNC and the polymer matrix (GPa)}$

Superscripts and Subscripts

- $c = \text{PMNC}$
- $f = \text{carbon fiber}$
- $in = \text{CNT/polymer interphase}$
- $m = \text{polymer material}$
- $n = \text{CNT}$
- $nc = \text{unwound PMNC containing straight CNTs}$
- $pf = \text{imaginary carbon fiber made of the polymer material}$
- $pm = \text{imaginary SCFF made of the polymer material}$

Acronyms

- CNT = carbon nanotube
- CVD = chemical vapor deposition
- FE = finite element
- MT = Mori–Tanaka
- NRLC = nanoreinforced laminated composite
- PMNC = polymer matrix nanocomposite
- SFFRC = short fuzzy fiber reinforced composite

Transactions of the ASME
Appendix: Explicit Forms of Constant Coefficients

The constants \((A_i)\) obtained in the course of deriving the shear lag model in Sec. 4 are explicitly expressed as follows:

\[
A_1 = \frac{a}{C_{60}(b^2 - a^2)} \left[ b^2 \ln \frac{r}{a} - \frac{(r^2 - a^2)}{2} \right], \quad A_2 = -\frac{b}{C_{60}(b^2 - a^2)} \left[ a^2 \ln \frac{r}{a} - \frac{(r^2 - a^2)}{2} \right]
\]

\[
A_3 = \frac{a}{C_{60}(b^2 - a^2)} \left[ b^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2} \right]
\]

\[
A_4 = \frac{b}{C_{60}(R^2 - b^2)} \left[ R^2 \ln \frac{R}{b} - \frac{(R^2 - b^2)}{2} \right] - \frac{b}{C_{60}(b^2 - a^2)} \left[ a^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2} \right]
\]

\[
A_5 = b^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2}, \quad A_6 = a^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2}
\]

\[
A_7 = \frac{b}{C_{60}(b^2 - a^2)} \left[ a^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2} \right]
\]

\[
A_8 = \frac{b}{C_{60}(R^2 - b^2)} \left[ R^2 \ln \frac{R}{b} - \frac{(R^2 - b^2)}{2} \right] - \frac{b}{C_{60}(b^2 - a^2)} \left[ a^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2} \right]
\]

\[
A_9 = -\frac{2a^3 C_{12}}{C_{11} b^3} + \frac{1}{C_{11} b^2}, \quad A_{10} = \frac{2b^2 C_{12}}{b^3 - a^3}, \quad A_{11} = \frac{2C_{12}}{b^3 - a^3}
\]

\[
A_{12} = \frac{a C_{11}}{C_{60}(b^2 - a^2)^2} \left[ a^2 b^2 + b^4 \ln \frac{b}{a} - \frac{a^4}{4} - \frac{3b^4}{4} \right]
\]

\[
A_{13} = -\frac{b C_{11}}{C_{60}(b^2 - a^2)^2} \left[ a^2 b^2 \ln \frac{b}{a} + \frac{a^4}{4} + \frac{b^4}{4} \right], \quad A_{14} = \frac{C_{11}}{C_{11}} + A_9 b_{14} + A_{10} b_{21} + A_{11} b_{31}
\]

\[
A_{15} = A_9 b_{14} + A_{10} b_{21} + A_{11} b_{31} + A_{12}, \quad A_{16} = A_9 b_{13} + A_{10} b_{23} + A_{11} b_{33} + A_{13}
\]

\[
A_{17} = A_9 b_{14} + A_{10} b_{24} + A_{11} b_{34}, \quad A_{18} = A_9 b_{15} + A_{10} b_{25} + A_{11} b_{35}
\]

\[
A_{19} = \frac{1}{C_{60}(R^2 - b^2)^2} \left[ b^2 R^2 + R^4 \ln \frac{R}{b} - \frac{b^4}{4} - \frac{3R^4}{4} \right] - \frac{1}{C_{60}(b^2 - a^2)^2} \left[ a^2 \ln \frac{b}{a} - \frac{(b^2 - a^2)}{2} \right]
\]

\[
A_{20} = \frac{C_{11}}{C_{11}} \frac{2C_{12}}{C_{12}} b_{11} + \frac{C_{11}}{C_{11}} b_{21}, \quad A_{21} = -\frac{2C_{11}}{C_{11}} b_{12} + \frac{2C_{12}}{C_{12}} b_{22} + \frac{C_{11}}{C_{11}} A_{13}
\]

\[
A_{22} = -\frac{2C_{11}}{C_{11}} b_{13} + \frac{C_{11}}{C_{11}} b_{23} + \frac{C_{11}}{C_{11}} A_{19}, \quad A_{23} = -\frac{2C_{11}}{C_{11}} b_{14} + \frac{2C_{12}}{C_{12}} b_{24}
\]

\[
A_{24} = -\frac{2C_{11}}{C_{11}} b_{15} + \frac{C_{11}}{C_{11}} \left[ \frac{R^2}{R^2 - b^2} \right] \left[ R^2 \ln \frac{R}{b} - \frac{(R^2 - b^2)}{2} \right] + \frac{C_{11}}{C_{11}} b_{25}
\]

\[
A_{25} = a^2 + \frac{(b^2 - a^2)}{2} A_{14} + (R^2 - b^2) A_{20}, \quad A_{26} = -(a/2) \left[ (b^2 - a^2) A_{15} + (R^2 - b^2) A_{21} \right]
\]

\[
A_{27} = \frac{R^2 - b^2}{2b} \left[ (b^2 - a^2) A_{16} + (R^2 - b^2) A_{22} \right], \quad A_{28} = (b^2 - a^2) A_{17} + (R^2 - b^2) A_{23}
\]

\[
A_{29} = (b^2 - a^2) A_{18} + (R^2 - b^2) A_{24}, \quad A_{30} = -\frac{a}{2} \left[ A_{15} A_{22} A_{16} - A_{21} \right]
\]

\[
A_{31} = \frac{1}{A_{30}} \left[ A_{14} A_{22} A_{16} - A_{20} + \left( \frac{a^2}{b^2 - a^2} \right) A_{22} A_{16} - A_{21} \right] \left( \frac{R^2 - b^2}{b^2 - a^2} \right) A_{22} A_{25} A_{27}
\]

\[
A_{32} = -\frac{1}{A_{30}} \left[ \left( \frac{R^2 - b^2}{b^2 - a^2} \right) A_{22} A_{25} A_{27} + A_{23} \right], \quad A_{33} = -\frac{1}{A_{30}} \left[ \left( \frac{R^2 - b^2}{b^2 - a^2} \right) A_{22} A_{25} A_{27} + \frac{R^2}{A_{27}} \right]
\]

\[
A_{34} = -\frac{1}{A_{30}} \left[ \frac{R^2 - b^2}{b^2 - a^2} \right] A_{22} A_{25} A_{27} + A_{28}, \quad A_{35} = -\frac{1}{A_{30}} \left[ \frac{R^2 - b^2}{b^2 - a^2} \right] A_{22} A_{25} A_{27} + A_{28}
\]

References


