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Control of Large Amplitude Vibrations of Doubly Curved Sandwich Shells Composed of Fuzzy Fiber Reinforced Composite Facings

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Abstract

This paper is concerned with the analysis of active constrained layer damping (ACLD) of geometrically nonlinear vibrations of doubly curved sandwich shells with facings composed of fuzzy fiber reinforced composite (FFRC). FFRC is a novel composite where the short carbon nanotubes (CNTs) which are either straight or wavy are radially grown on the periphery of the long continuous carbon fiber reinforcements. The plane of waviness of the CNTs is coplanar with the plane of carbon fiber. The constraining layer of the ACLD treatment is composed of the vertically/obliquely reinforced 1-3 piezoelectric composites (PZCs) while the constrained viscoelastic layer has been sandwiched between the substrate and the PZC layer. The Golla–Hughes–McTavish (GHM) method has been implemented to model the constrained viscoelastic layer of the ACLD treatment in time domain. A three dimensional finite element (FE) model of smart doubly curved FFRC sandwich shells integrated with ACLD patches has been developed to investigate the performance of these patches for controlling the geometrically nonlinear vibrations of these shells. This study reveals that the performance of the ACLD patches for controlling the geometrically nonlinear vibrations of the doubly curved sandwich shells is better in the case of the facings composed of laminated FFRC than that in the case of the facings made of conventional orthotropic laminated composite. The research carried out in this paper brings to light that even the wavy CNTs can be properly utilized for attaining structural benefits from the exceptional elastic properties of CNTs.

Keywords: Fuzzy fiber, 1-3 Piezoelectric composite (PZC), Smart doubly curved Sandwich shells, active constrained layer damping (ACLD), nonlinear vibrations.

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1. Introduction

A typical sandwich structure is a three layered material system which consists of two thin but stiff facings that are separated by a light weight thick flexible core [1-6]. Sandwich structures used in several applications such as in aerospace, automobile, locomotive and construction industries have shown to possess improved fatigue strength, acoustical insulation and the capability of absorbing mechanical energy when compared to the conventional monocoque structures. The thin facings are usually stiff which are made up of materials having high tensile and compressive strength while the thick core is composed of low density material capable of carrying an appreciable amount of transverse load analogous to that of an I-beam [2, 5]. The function of the core is to distance the facings away from the neutral axis, henceforth enhancing the value of the moment of inertia resulting in high section modulus of the structure. This type of structural construction where the direct bending loads are carried by the facings and the shear forces are carried by the core is capable of resisting bending and buckling loads [1]. Owing to the above features, the strength and stiffness of the sandwich construction are greatly enhanced without a corresponding increase in weight [5].

The research on the synthesis of molecular carbon structure by an arc-discharge method for evaporation of carbon, led to the discovery of an extremely thin needle-like graphitic carbon nanotube [7]. Many research investigations revealed that the CNTs have axial Young’s modulus in the terapascal range [8, 9]. The quest for utilizing such exceptional mechanical properties of CNTs and their high aspect ratio and low density led to the opening of an emerging area of research on the development of CNT-reinforced nanocomposites [10, 11]. It has been experimentally observed that CNTs are actually curved cylindrical tubes with a relatively high aspect ratio [12–14]. It is hypothesized that their affinity to become curved is due to their high aspect ratio and the associated low bending stiffness. Tsai et al. [15] studied the effects of CNT waviness and its distribution on the effective nanocomposite stiffness. For structural applications, the manufacturing of two-phase unidirectional continuous CNT-reinforced composites in large scale has to encounter some challenging difficulties. Typical among these are the agglomeration of CNTs, the misalignment and the difficulty in manufacturing long CNTs.

Alignment, dispersion and adhesion of CNTs in polymer matrices are vital for structural applications of CNTs. More success can be achieved for improving the transverse
multifunctional properties of the hybrid CNT-reinforced composites by growing the short CNTs on the circumferential surfaces of the advanced fibers [16–20]. A fiber coated with radially grown CNTs on its circumferential surface is being called as “fuzzy fiber” [18, 20] and the resulting composite may be called as fuzzy fiber reinforced composite (FFRC) [20-22]. Recently, extensive studies on the prediction of effective properties of FFRC containing straight CNTs have been reported by Kundalwal and Ray [20, 21] using micromechanical analysis and the finite element (FE) method. Most recently, Kundalwal and Ray [22] reported that the wavy CNTs can be properly grown on the circumferential surfaces of the carbon fibers to improve the in-plane effective elastic properties of the continuous FFRC.

Over the past few decades, smart structures [23-26] have been the focus of the study of many researchers with the aim intended at passive for purpose and adaptive for crisis like situations. These structures due to these abilities find application in the civil constructions which are prone to earth quakes, aircrafts during landing and takeoff and automobiles during crashes. Active constrained layer damping (ACLD) treatment developed by Baz [27] has demonstrated such adaptability and is found very efficient in health monitoring of structures. ACLD treatment consists of a constrained viscoelastic layer sandwiched between the host structure and the constraining piezoelectric material. ACLD offers the attributes of both active and passive damping depending on whether the constraining layer is deactivated (voltage \(V = 0\)) or activated (voltage \(V \neq 0\)). With the advent of piezoelectric composites (PZC), the ACLD treatment gained popularity and has been found more reliable for the efficient control of flexible structures. In PZCs, the reinforcements are made of the existing monolithic piezoelectric materials and the matrix is the conventional epoxy. Among the various PZC materials commercially available till today, laminae of the vertically/obliquely reinforced 1–3 PZC materials [28, 29] are found to possess wide range of effective properties and are being effectively used in medical imaging applications and high frequency underwater transducers. The constructional feature of a layer of the vertically/obliquely reinforced 1–3 PZC material is illustrated in Fig. 1(a) where the reinforcing piezoelectric fibers are coplanar with the vertical \(xz\)- or \(yz\)-plane making an angle \(\psi\) with respect to the \(z\)-axis while the fibers are poled along their length. For the vertically reinforced 1–3 PZC the value of \(\psi\) is zero and in case of the obliquely reinforced 1–3 PZC its value is nonzero.
Flexible composite sandwich structures are prone to undergo finite amplitude vibrations due to their low internal damping. Various studies on large amplitude free vibrations of composite structures and sandwich structures using approximate analytical and finite element methods have been reported by many researchers [30-36]. However, very few studies concerning the geometrically nonlinear dynamic analysis of thin smart flexible sandwich structures are available in the open literature. Tzou and Zhou [37] investigated the dynamic control of nonlinear circular plates/shells composed of two surface piezoelectric layers and one isotropic elastic layer. Yi et al. [38] presented a nonlinear dynamic analysis of structures integrated with piezoelectric sensors/actuators. Ray and Baz [39] developed a variational model to investigate the control of nonlinear vibrations of beams using ACLD treatment. Gao and Shen [40] showed by finite element (FE) analysis that the piezoelectric actuators can significantly suppress the geometrically nonlinear transient vibrations of composite plates. Shen [41] presented a nonlinear bending analysis for simply-supported, shear deformable antisymmetric cross-ply laminated plates integrated with piezoelectric actuators and subjected to combined action of mechanical, electrical and thermal loads. Lentzen and Schmidt [42] developed a nonlinear FE model for static and dynamic analyses of smart composite structures integrated with piezoelectric layers. Huang et al. [43] analyzed nonlinear dynamic responses of simply-supported shear deformable cross-ply laminated plates integrated with piezoelectric actuators. Although the piezoelectric actuator layer acts more efficiently as the constraining layer of the ACLD treatment, researchers did not pay much attention to the use of the ACLD treatment for controlling the nonlinear vibrations of composite sandwich structures. Recently, Sarangi and Ray [44, 45] carried out ACLD of geometrically nonlinear vibrations of laminated composite shallow shells and doubly curved shells using vertically/obliquely reinforced 1-3 PZC and active fiber composite (AFC) material as the constraining layer. Most recently, Kumar and Ray [46, 47] analyzed active control of geometrically nonlinear vibrations of composite sandwich plates and doubly curved composite sandwich shells using 1-3 PZC material.

In order to establish the credibility of using CNTs for structural applications, extensive research on the analysis of CNT-reinforced composite structures is needed and is not yet reported. It may be difficult to manufacture continuous CNT-reinforced composite shell of substantial thickness for structural applications as long CNTs are difficult to manufacture. Using
FFRC, thin stiff laminae may be constructed without facing much manufacturing challenges. Thus, the analysis of smart doubly curved sandwich shell with its facings being composed of thin layers of FFRC may be an important study for investigating the structural benefits from the use of CNTs. Also, the dynamic characteristics of the doubly curved sandwich shells with its facings being composed of FFRC may be different from that of the conventional orthotropic laminated doubly curved composite/sandwich shells. Hence, the necessity for further study on the ACLD of geometrically nonlinear vibrations of doubly curved FFRC sandwich shells arises. However, no work is reported yet on the ACLD of geometrically nonlinear vibrations of doubly curved FFRC sandwich shells using vertically/obliquely reinforced 1–3 PZC materials.

In this paper, authors intend to investigate the active damping of geometrically nonlinear vibrations of smart doubly curved FFRC sandwich shells using vertically/obliquely reinforced 1-3 PZC material. The top and the bottom faces of the substrate sandwich shell are composed of N number of orthotropic FFRC layers while the core is made of a soft flexible honeycomb material. Using layerwise First order shear deformation theory (FSDT) and satisfying the interfacial continuity conditions, a three dimensional finite element model has been developed for the FFRC sandwich shells integrated with the centrally mounted patch of the ACLD treatment. The constrained viscoelastic layer of the ACLD treatment is assumed to be linearly viscoelastic and has been modeled using the Golla–Hughes–McTavish (GHM) method [48-50]. Several substrate FFRC sandwich shells with different fiber volume fractions of orthotropic laminated FFRC facings containing straight and wavy CNTs are considered for presenting the numerical results. The nonlinear fundamental frequency ratios of the doubly curved FFRC sandwich shells with different facing configurations have been estimated. Also, the effect of variation of the piezoelectric fiber orientation angle ($\psi$) in the 1–3 PZC constraining layer on the control authority of the ACLD patch has been investigated.

2. Architecture of a Novel Fuzzy Fiber Reinforced Composite

Figure 1 (b) illustrates the schematic sketch of a lamina of the continuous FFRC containing the fuzzy fibers which are formed by radially growing sinusoidally wavy CNTs on the surface of the continuous carbon fibers. The plane of waviness of the CNTs is considered to be coplanar with the longitudinal plane of the carbon fiber. The fuzzy fibers containing straight
CNTs and wavy CNTs being coplanar with the longitudinal plane of the carbon fiber are shown in Figs. 2 (a) and (b), respectively. The orientation of the plane of the wavy CNTs radially grown on the circumferential surface of the carbon fiber is an important issue because the planer orientations of the wavy CNTs may influence the elastic properties of the FFRC. Kundalwal and Ray [30] considered two possible planar orientations of the wavy CNTs i.e., the wavy CNTs are coplanar with either the longitudinal plane (i.e., 1–3) or the transverse plane (i.e., 2–3) of the carbon fiber to investigate the effect of wavy CNTs on the effective elastic properties of the FFRC. They reported that when the wavy CNTs are coplanar with the longitudinal plane of the carbon fibers, the axial elastic coefficients of the FFRC are significantly improved over those of the FFRC with straight CNTs. Hence, in this study the wavy CNTs are considered to be coplanar with the longitudinal plane (i.e., 1–3 or 1’–3’ plane) of the carbon fiber as shown in Fig. 2 (b). A representative volume element (RVE) of the CNT-reinforced nanocomposite containing a wavy CNT has been considered by Kundalwal and Ray [30] to predict the effective elastic properties of the FFRC and such an RVE is schematically illustrated in Fig. 3. The CNT waviness is assumed to be sinusoidal in the longitudinal plane of the carbon fiber and is defined by

\[
x = A \sin(\omega y);
\]

\[
\omega = n\pi/L_n
\]

in which A and L_n are the amplitude of the CNT wave and the linear distance between the CNT ends, respectively, while n represents the number of CNT waves. The running length L_{rw} of the CNT is given by:

\[
L_{rw} = \int_0^L \sqrt{1 + A^2 \omega^2 \cos^2(\omega y)} \, dy
\]

where the angle \(\theta\) shown in Fig. 3 is given by

\[
\tan \theta = \frac{dx}{dy} = A\omega \cos(\omega y)
\]

3. Governing Equations

Figure 4 illustrates a schematic diagram of a smart doubly curved FFRC sandwich shell. The substrate of this smart structure is a sandwich shell. The top and the bottom faces of the sandwich shell are composed of ‘N’ number of unidirectional orthotropic FFRC layers while the core is a flexible isotropic HEREX honeycomb (soft) structure. The top surface of the top facing of the substrate shell is integrated with a centrally mounted patch of the ACLD treatment. The constraining layer of the ACLD treatment is composed of the vertically/obliquely reinforced 1-3
PZC layer. The curvilinear length, the curvilinear width and the thickness of the shell are denoted by $a$, $b$ and $H$, respectively. The thickness of the constraining PZC layer and the constrained viscoelastic layer of the ACLD treatment are $h_p$ and $h_v$, respectively. The thicknesses of each facing and half of the thickness of the core are $h$ and $h_c \left(2h_c = h\right)$, respectively. For the purpose of modeling, the core is assumed to be an equivalent homogeneous solid continuum [4, 6]. The elastic properties of the layers of the overall shell differ in orders hence, a single displacement theory cannot be used to describe the kinematics of deformations of the overall structures. The kinematics of deformations of the transverse normal in the $xz$- or $yz$- planes, have been illustrated in Figs. 5(a) and 5(b), respectively. $\theta, \phi, \alpha, \beta$ and $\gamma$ represent the rotations of the portions of the transverse normal lying in the core, the top face, the bottom face, the viscoelastic layer and the piezoelectric layer, respectively in the $xz$-plane while $\theta_2, \phi_2, \alpha_2, \beta_2$ and $\gamma_2$ represent the same in the $yz$-plane. The gradients of the transverse normal displacements at any point in the core, the top face, the bottom face, the viscoelastic layer and the piezoelectric layer are represented by the variables $\theta_i, \phi_i, \alpha_i, \beta_i$ and $\gamma_i$. The displacement fields $u_i^t, u_i^b, u_i^c, u_i^v$ and $u_i^p (i = 1, 2$ and $3)$ describing the kinematics of deformations while satisfying the continuity of displacements at the interface between the adjacent layers along $x, y$ and $z$ directions, respectively for the top face, the bottom face, the core, the constrained viscoelastic layer and the 1-3 PZC constraining layer can be expressed as follows:

$$
\begin{align*}
\mathbf{u}_b^t(x, y, z, t) &= \mathbf{u}_{0t}(x, y, t) + h_c \theta_t(x, y, t) + (z - h_c)\alpha_t(x, y, t) \\
\mathbf{u}_b^b(x, y, z, t) &= \mathbf{u}_{0b}(x, y, t) - h_c \theta_b(x, y, t) + (z + h_c)\phi_b(x, y, t) \\
\mathbf{u}_i^c(x, y, z, t) &= \mathbf{u}_{0i}(x, y, t) + h_c \theta_c(x, y, t) \\
\mathbf{u}_i^v(x, y, z, t) &= \mathbf{u}_{0i}(x, y, t) + h_c \theta_i(x, y, t) + h_\alpha_i(x, y, t) + (z - h_4)\beta_i(x, y, t) \\
\mathbf{u}_i^p(x, y, z, t) &= \mathbf{u}_{0i}(x, y, t) + h_c \theta_i(x, y, t) + h_\alpha_i(x, y, t) + h_v\beta_i(x, y, t) + (z - h_5)\gamma_i(x, y, t)
\end{align*}
$$

In Eq. (4), $\mathbf{u}_{0i} (i = 1, 2$ and $3)$ denote the translational displacements at any point on the mid-plane of the core along $x, y$ and $z$ directions, respectively. The superscripts $tf$, $bf$, $c$, $v$ and $p$
represent the top facing, bottom facing, the central core, the viscoelastic layer and the piezoelectric layer, respectively. Note that these are also used as subscripts elsewhere. It is obvious from the above displacement fields that the displacements are continuous at the interface between two adjacent continua. For the ease of analysis and computation, the translational displacement variables are separated from the other variables as follows:

\[
\{d_i\} = [u_{01} \quad u_{02} \quad u_{03}]^T \quad \text{and} \quad \{d_r\} = \begin{bmatrix} \theta \\ \phi \\ \alpha \\ \beta \\ \gamma \end{bmatrix}^T
\]

where \( \{\theta\} = [\theta_1 \quad \theta_2 \quad \theta_3], \{\phi\} = [\phi_1 \quad \phi_2 \quad \phi_3], \{\alpha\} = [\alpha_1 \quad \alpha_2 \quad \alpha_3], \{\beta\} = [\beta_1 \quad \beta_2 \quad \beta_3] \) and \( \{\gamma\} = [\gamma_1 \quad \gamma_2 \quad \gamma_3] \)

The state of strain at any point in the overall shell is divided into the following two strain vectors \( \{\epsilon_b\} \) and \( \{\epsilon_c\} \)

\[
\{\epsilon_b\} = \{\epsilon_1 \quad \epsilon_2 \quad \epsilon_{12} \quad \epsilon_3\}^T \quad \text{and} \quad \{\epsilon_c\} = \{\epsilon_{13} \quad \epsilon_{23}\}^T
\]

where \( \epsilon_i (i = 1, 2 \text{ and } 3) \) are the normal strains along \( x, y \) and \( z \) directions, respectively; \( \epsilon_{12} \) is the in-plane shear strain and \( \epsilon_{13}, \epsilon_{23} \) are the transverse shear strains. The superscripts \( b \) and \( s \) denote bending and transverse shear, respectively. By using the displacement fields and the Green-Lagrange type nonlinear strain-displacement relations, the strain vectors at any point in the core, the face sheets, the viscoelastic layer and the active constraining layer, respectively, can be expressed as:

\[
\{\epsilon_b\}_c = \{\epsilon_{bt}\} + \{Z_1\}\{\epsilon_{br}\} + \{\epsilon_{bnr}\}, \quad \{\epsilon_b\}_b = \{\epsilon_{bt}\} + \{Z_2\}\{\epsilon_{br}\} + \{\epsilon_{bnr}\} \\
\{\epsilon_b\}_t = \{\epsilon_{bt}\} + \{Z_3\}\{\epsilon_{br}\} + \{\epsilon_{bnr}\}, \quad \{\epsilon_b\}_p = \{\epsilon_{bt}\} + \{Z_4\}\{\epsilon_{br}\} + \{\epsilon_{bnr}\} \\
\{\epsilon_s\}_c = \{\epsilon_{st}\} + \{Z_5\}\{\epsilon_{sr}\} + \{\epsilon_{snr}\}, \quad \{\epsilon_s\}_b = \{\epsilon_{st}\} + \{Z_6\}\{\epsilon_{sr}\} + \{\epsilon_{snr}\} \\
\{\epsilon_s\}_t = \{\epsilon_{st}\} + \{Z_7\}\{\epsilon_{sr}\} + \{\epsilon_{snr}\}, \quad \{\epsilon_s\}_v = \{\epsilon_{sv}\} + \{Z_8\}\{\epsilon_{sr}\} + \{\epsilon_{snr}\} \quad \text{and} \\
\{\epsilon_s\}_p = \{\epsilon_{st}\} + \{Z_9\}\{\epsilon_{sr}\} + \{\epsilon_{snr}\}
\]

in which \( \{\epsilon_{bt}\}, \{\epsilon_{br}\}, \{\epsilon_{bnr}\} \) and \( \{\epsilon_{bt}\}_p \) represent the in-plane and transverse normal strain vectors while \( \{\epsilon_{st}\}, \{\epsilon_{sr}\}, \{\epsilon_{snr}\} \) and \( \{\epsilon_{sv}\}_p \) represent the transverse strain vectors. The various matrices appearing in the above equations have been defined in the Appendix, while the generalized strain vectors appearing in Eq. (7) are given by
\{\varepsilon_{bt}\} = \nabla_1 \{d_1\} \text{ and } \{\varepsilon_{st}\} = \nabla_2 \{d_1\} \hspace{1cm} (8)

\{\varepsilon_{bt}\} = \begin{bmatrix} \nabla_3 [\theta]^T & \nabla_3 [\phi]^T & \nabla_3 [\alpha]^T & \nabla_3 [\beta]^T & \nabla_3 [\gamma]^T \end{bmatrix}^T \hspace{1cm} (9)

\{\varepsilon_{st}\} = \begin{bmatrix} \nabla_4 [\theta]^T & \nabla_4 [\phi]^T & \nabla_4 [\alpha]^T & \nabla_4 [\beta]^T & \nabla_4 [\gamma]^T \end{bmatrix}^T \hspace{1cm} (10)

\{\varepsilon_{sw}\} = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u_{o1}}{\partial x}\right)^2 & \left(\frac{\partial u_{o1}}{\partial y}\right)^2 & \left(\frac{\partial u_{o2}}{\partial x}\right)^2 & 2 \left(\frac{\partial u_{o1}}{\partial x}\right) \left(\frac{\partial u_{o1}}{\partial y}\right) & \left(\frac{\partial u_{o3}}{\partial x}\right)^2 & \left(\frac{\partial u_{o3}}{\partial y}\right)^2 & \left(\frac{\partial u_{o2}}{\partial z}\right)^2 & 2 \left(\frac{\partial u_{o2}}{\partial x}\right) \left(\frac{\partial u_{o2}}{\partial z}\right) & \left(\frac{\partial u_{o3}}{\partial z}\right)^2 & \left(\frac{\partial u_{o3}}{\partial y}\right)^2 & \left(\frac{\partial u_{o2}}{\partial z}\right)^2 & 2 \left(\frac{\partial u_{o2}}{\partial x}\right) \left(\frac{\partial u_{o2}}{\partial y}\right) & \left(\frac{\partial u_{o3}}{\partial x}\right)^2 & \left(\frac{\partial u_{o3}}{\partial y}\right)^2 & \left(\frac{\partial u_{o2}}{\partial z}\right)^2 \end{bmatrix}^T \hspace{1cm} (11)

\begin{align*}
\nabla_1 &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{1}{R_1} \\ 0 & \frac{\partial}{\partial y} & \frac{1}{R_2} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, & \nabla_2 &= \begin{bmatrix} -\frac{1}{R_1} & 0 & \frac{\partial}{\partial x} \\ 0 & -\frac{1}{R_2} & \frac{\partial}{\partial y} \\ 0 & 0 & 1 \end{bmatrix}, & \nabla_3 &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}, & \nabla_4 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}.
\end{align*}

In consistent with the state of strains given by Eq. (6), the state of stresses at any point in the overall shell is described by the two stress vectors as follows:

\begin{align*}
\{\sigma_t\} &= \{\sigma_1, \sigma_2, \sigma_{12}, \sigma_3\}^T \text{ and } \{\sigma_s\} = \{\sigma_{13}, \sigma_{23}\}^T \hspace{1cm} (12)
\end{align*}

where, \(\sigma_i (i = 1, 2, 3)\) are the normal stresses along \(x, y\) and \(z\) directions, respectively; \(\sigma_{12}\) is the in-plane shear stress; \(\sigma_{13}, \sigma_{23}\) are the transverse shear stresses. The constitutive relation for the materials of the different layers of the top and the bottom face of the sandwich shell and that of the flexible core are given by

\begin{align*}
\{\sigma_{b}^k\}_t &= \left[ C_{b}^k \right] \{\varepsilon_{b}^k\}_t, \quad \{\sigma_{b}^k\}_b = \left[ C_{b}^k \right] \{\varepsilon_{b}^k\}_b \quad \text{and} \quad \{\sigma_{b}^k\}_\text{core} = \left[ C_{b}^k \right] \{\varepsilon_{b}^k\}_\text{core} \\
\{\sigma_{s}^k\}_t &= \left[ C_{s}^k \right] \{\varepsilon_{s}^k\}_t, \quad \{\sigma_{s}^k\}_b = \left[ C_{s}^k \right] \{\varepsilon_{s}^k\}_b \quad \text{and} \quad \{\sigma_{s}^k\}_\text{core} = \left[ C_{s}^k \right] \{\varepsilon_{s}^k\}_\text{core}; \quad (k = 1, 2, 3...N) \hspace{1cm} (13)
\end{align*}

where
\begin{align*}
\left[ C_{b}^k \right] &= \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}, & \left[ C_{s}^k \right] &= \begin{bmatrix} C_{55} & C_{45} \\ C_{45} & C_{44} \end{bmatrix}.
\end{align*}
with \( \overline{C}_{ij} \) (i, j = 1, 2, 3...6) being the transformed elastic coefficients of the laminated FFRC facings with respect to the reference co-ordinate system. The present study is concerned with the analysis of the laminated doubly curved FFRC sandwich shells undergoing ACLD in the time domain. As the constrained layer damping of the host structure is attributed to the transverse shear deformations of the viscoelastic constrained layer and the extensional stiffness of the viscoelastic constrained layer is very small as compared to the overall FFRC substrate sandwich shell, the extensional strain energy of the constrained viscoelastic layer may be neglected. Thus, the constitutive relations for the homogeneous isotropic linear viscoelastic material layer can be written as

\[
\{\sigma_s\}_v = G(t)\{\epsilon_s(0)\}_v + \int_0^t G(t - \tau) \frac{\partial\{\epsilon_s(\tau)\}_v}{\partial \tau} d\tau
\]

where \( G(t) \) is the time-dependent relaxation function and \( \epsilon(t) \) is confined to zero in the limits \( t \in (-\infty, 0) \). In the time domain, considering the above constraints, the viscoelastic material constitutive relation given by Eq. (20) reduces to the following Stieltjes integral form [49]

\[
\{\sigma_s\}_v = \int_0^t G(t - \tau) \frac{\partial\{\epsilon_s\}_v}{\partial \tau} d\tau
\]

The constitutive relations for the constraining 1-3 PZC layer of the ACLD treatment compatible with the present method of finite element formulation are given by [27]

\[
\{\sigma_b\}_p = [C_b]_p \{\epsilon_b\}_p + [C_{bs}]_p \{\epsilon_s\}_p - \{\epsilon_b\}E_z, \quad \{\sigma_s\}_p = [C_{bs}]_p^T \{\epsilon_b\}_p + [C_s]_p \{\epsilon_s\}_p - \{\epsilon_s\}E_z \quad \text{and} \quad D_z = \{e_b\}^T \{\epsilon_b\}_p + \{e_s\}^T \{\epsilon_s\}_p + \epsilon_{33}E_z
\]

Here, \( E_z \) and \( D_z \) represent the electric field and the electric displacement along the \( z \)-direction, respectively and \( \epsilon_{33} \) is the dielectric constant. The forms of the transformed elastic coefficient matrices \([C_b]_p\) and \([C_s]_p\) are similar to those of \([C_b]\) and\([C_s]\), respectively. It may be noted
from the above form of constitutive relations that the transverse shear strains are coupled with the in-plane stresses due to the orientation of piezoelectric fibers in the vertical $xz$ - or $yz$ - plane and corresponding coupling elastic constant matrix $[C_{bs}]_p$ is given by

\[
[C_{bs}]_p = \begin{bmatrix}
C_{15} & 0 \\
C_{25} & 0 \\
0 & C_{46} \\
C_{35} & 0
\end{bmatrix}
\quad \text{or} \quad
[C_{bs}]_p = \begin{bmatrix}
0 & C_{14} \\
0 & C_{24} \\
C_{56} & 0 \\
0 & C_{34}
\end{bmatrix}
\]

(18)

according as the piezoelectric fibers are coplanar with the vertical $xz$ - or $yz$ - plane. Note that if the fibers are coplanar with both the $xz$ - and the $yz$ - plane, this coupling matrix becomes a null matrix. Also, the piezoelectric constant matrices $\{e_b\}$ and $\{e_s\}$ appearing in Eq. (17) contain the following transformed effective piezoelectric coefficients of the 1-3 PZC:

\[
\{e_b\} = \{\overline{e}_{31} \, \overline{e}_{32} \, \overline{e}_{36} \, \overline{e}_{33} \}^T \quad \text{and} \quad \{e_s\} = \{\overline{e}_{35} \, \overline{e}_{34} \}^T
\]

(19)

The total potential energy $T_p$ and the kinetic energy $T_k$ of the overall shell/ACLD system are given by

\[
T_p = \frac{1}{2} \int \left[ \int \{e_b\}^T \{\sigma_b\} + \{e_s\}^T \{\sigma_s\} \right] d\Omega + \sum_{k=1}^{N} \int \left[ \{e_b^k\}^T \{\sigma_b^k\} + \{e_s^k\}^T \{\sigma_s^k\} \right] d\Omega
\]

\[
+ \sum_{k=1}^{N} \int \left[ \{e_b^k\}^T \{\sigma_b^k\} + \{e_s^k\}^T \{\sigma_s^k\} \right] d\Omega + \int \left[ \{e_b\}^T \{\sigma_b\} + \{e_s\}^T \{\sigma_s\} \right] d\Omega
\]

\[
+ \int \left[ \{e_b\} \right]^T \left[ \{e_s\} \right] d\Omega - \int D_z E_z d\Omega - \int \{d\}^T \{f\} dA
\]

(20)

\[
T_k = \frac{1}{2} \int \rho_c [(\dot{u}^c)^2 + (\dot{u}^c)^2 + (\dot{u}^c)^2] d\Omega + \int \rho_s [(\dot{u}^s)^2 + (\dot{u}^s)^2] d\Omega
\]

\[
+ \int \rho_p [(\dot{u}^p)^2 + (\dot{u}^p)^2 + (\dot{u}^p)^2] d\Omega + \sum_{k=1}^{N} \rho_b^k [(\dot{u}^{bf})^2 + (\dot{u}^{bf})^2 + (\dot{u}^{bf})^2] d\Omega
\]

\[
+ \sum_{k=1}^{N} \rho_s^k [(\dot{u}^{sf})^2 + (\dot{u}^{sf})^2 + (\dot{u}^{sf})^2] d\Omega
\]

(21)

in which $\rho_c$, $\rho_v$ and $\rho_p$ are the mass densities of the core, the viscoelastic layer and the 1-3 PZC layers, respectively. $\rho_b^k$ and $\rho_s^k$ are the densities of the $k^{th}$ layer of the bottom and the top FFRC face sheets, respectively. $\{f\} = [0 \quad 0 \quad P]^T$ is the externally applied surface traction acting over a
surface area $A$ with $P$ being the transverse distributed pulse loading over the surface and represents the $\Omega$ volume of the concerned layer.

4. Finite Element Model of doubly curved FFRC sandwich shell integrated with the ACLD patch

The overall shell is discretized by the eight noded isoparametric quadrilateral elements. Following Eq. (5), the generalized displacement vectors $\{d_{ij}^e\}$ and $\{d_{rij}^e\}$ associated with the $j^{th}$ node of the element can be written as

$$\{d_{ij}^e\} = [u_{0ij} \quad u_{02j} \quad u_{03j}]^T$$

and

$$\{d_{rij}^e\} = [\theta_{ij} \quad \theta_{3j} \quad \phi_{ij} \quad \phi_{3j} \quad \alpha_{ij} \quad \alpha_{3j} \quad \beta_{ij} \quad \beta_{3j} \quad \gamma_{ij} \quad \gamma_{3j}]^T$$

(22)

Thus the generalized displacement vector at any point within the element can be expressed in terms of the nodal generalized displacement vectors $\{d_e\}$ and $\{d_r\}$ as follows:

$$\{d_e\} = [N] \{d_e^r\} \quad \text{and} \quad \{d_r\} = [N_r] \{d_r^e\}$$

(23)

in which

$$[N] = [N_{11}] \quad [N_{22}] \quad [N_{33}] \quad \ldots \quad [N_{88}] \quad [N_{nk}] = n_k I, \quad k = 1, 2, 3, \ldots, 8;$$

$$\{d_e\} = \left[ \{d_{11}^e\}^T \quad \{d_{22}^e\}^T \quad \ldots \quad \{d_{88}^e\}^T \right]^T \quad \text{and} \quad \{d_r\} = \left[ \{d_{11}^r\}^T \quad \{d_{22}^r\}^T \quad \ldots \quad \{d_{88}^r\}^T \right]^T$$

(24)

while $I$ is the (3 x 3) and the (15 x 15) identity matrices, respectively and $n_j$ is the shape function of natural coordinates associated with the $j^{th}$ node. Making use of the relation given by Eqs. (7)-(12), (23) and (24), the strain vectors at any point within the element can be expressed in terms of the nodal generalized displacement vectors as follows:

$$\{\varepsilon_{eb}\}_e = [B_{eb}] \{d_e\}_e + [Z_e] [B_{eb}] \{d_r\}_e + \frac{1}{2} [B_e] [B_e^T] \{d_e\}_e,$$

$$\{\varepsilon_{eb}\}_b = [B_{eb}] \{d_e\}_b + [Z_b] [B_{eb}] \{d_r\}_b + \frac{1}{2} [B_e] [B_e^T] \{d_e\}_b,$$

$$\{\varepsilon_{eb}\}_t = [B_{eb}] \{d_e\}_t + [Z_t] [B_{eb}] \{d_r\}_t + \frac{1}{2} [B_e] [B_e^T] \{d_e\}_t,$$

$$\{\varepsilon_{eb}\}_p = [B_{eb}] \{d_e\}_p + [Z_p] [B_{eb}] \{d_r\}_p + \frac{1}{2} [B_e] [B_e^T] \{d_e\}_p.$$
in which the nodal strain-displacement matrices $[B_{ib}]$, $[B_{ib}]$, $[B_{is}]$, $[B_{is}]$, $[B_{is}]$ and $[B_{is}]$ are given by

$$
[B_{ib}] = [B_{ib1}, \ldots, B_{ib8}], \quad [B_{ib}] = [B_{ib1}, \ldots, B_{ib8}],
$$

$$
[B_{is}] = [B_{is1}, \ldots, B_{is8}], \quad [B_{is}] = [B_{is1}, \ldots, B_{is8}],
$$

\[
[B_i] = \begin{bmatrix}
\frac{\partial w_0}{\partial x} & 0 & \frac{\partial w_0}{\partial x} & 0 \\
0 & \frac{\partial w_0}{\partial x} & 0 & 0
\end{bmatrix}^T \quad \text{and} \quad [B_z] = [B_{z1}, \ldots, B_{z8}]
\] (26)

The submatrices of $[B_{ib}]$, $[B_{ib}]$, $[B_{is}]$ and $[B_{is}]$ and shown in Eq. (26) have been explicitly presented in the Appendix. Finally, substituting Eqs. (13), (16)-(19) and (25) into Eq. (20) and recognizing that $E_z = -V/\varepsilon$ with $V$ being the applied voltage across the thickness of the piezoelectric layer, the total potential energy $T_p^e$ of a typical element integrated with the ACLD treatment can be expressed as

$$
T_p^e = \frac{1}{2} \left[ \{d_i^e\}^T [K_i^e] \{d_i^e\} + \{d_j^e\}^T [K_j^e] \{d_j^e\} + \{d_r^e\}^T [K_r^e] \{d_r^e\} + \{d_t^e\}^T [K_t^e] \{d_t^e\} \right. \\
+ \{d_j^e\}^T [K_j^e] \int_0^{t(t - \tau)} \frac{\partial}{\partial \tau} \{d_j^e\} d\tau + \{d_r^e\}^T [K_r^e] \int_0^{t(t - \tau)} \frac{\partial}{\partial \tau} \{d_r^e\} d\tau \\
+ \{d_r^e\}^T [K_r^e] \int_0^{t(t - \tau)} \frac{\partial}{\partial \tau} \{d_r^e\} d\tau + \{d_t^e\}^T [K_t^e] \int_0^{t(t - \tau)} \frac{\partial}{\partial \tau} \{d_t^e\} d\tau \\
- 2 \{d_i^e\}^T \{F_{ip}^e\} V - 2 \{d_i^e\}^T \{F_{ipn}^e\} V - 2 \{d_j^e\}^T \{F_{ip}^e\} V - \varepsilon_{33} \frac{V^2}{h_p} - 2 \{d_j^e\}^T \{F^e\} \right]
$$ (27)

The elemental stiffness matrices $[K_i^e]$, $[K_j^e]$, $[K_r^e]$, $[K_t^e]$, $[K_{jtr}^e]$, $[K_{jtr}^e]$ and $[K_{jtr}^e]$, the elemental electro-elastic coupling matrices $\{F_{ip}^e\}$, $\{F_{ipn}^e\}$ and $\{F_{ip}^e\}$ and the elemental load vector $\{F^e\}$ appearing in Eq. (27) are derived as follows:
\[
\begin{align*}
\{ F^e_{1p} \} &= \{ F^e_{1\alpha} \} + \{ F^e_{1\beta} \}, \quad \{ F^e_{1\gamma} \} = \{ F^e_{1\pi} \}, \\
\end{align*}
\]

where the stiffness matrices concerning bending deformations are

\[
\begin{align*}
[K^e_{1\alpha}] &= \int_0^b \int_0^b \left[ D_{1\alpha}^b \right]^T \left[ D_{1\alpha}^b \right] + \left[ D_{1\beta}^b \right]^T \left[ D_{1\beta}^b \right] + \left[ D_{1\gamma}^b \right]^T \left[ D_{1\gamma}^b \right] d\gamma d\beta \\
K^e_{1\beta} &= \int_0^b \int_0^b \left[ D_{1\alpha}^b \right]^T \left[ D_{1\beta}^b \right] + \left[ D_{1\beta}^b \right]^T \left[ D_{1\alpha}^b \right] d\gamma d\beta \\
K^e_{1\gamma} &= \int_0^b \int_0^b \left[ D_{1\beta}^b \right]^T \left[ D_{1\alpha}^b \right] + \left[ D_{1\alpha}^b \right]^T \left[ D_{1\beta}^b \right] d\gamma d\beta \\
K^e_{1\pi} &= \int_0^b \int_0^b \left[ D_{1\pi}^b \right]^T \left[ D_{1\pi}^b \right] d\gamma d\beta,
\end{align*}
\]

and those associated with the transverse shear deformations are
\[
\begin{align*}
[K^e_{ts}] &= \int_0^{a_t} \int_0^{b_t} \left( [B_{ts}]^T \left( [D^b_{ts}] + [D^c_{ts}] + [D^p_{ts}] \right) [B_{ts}] \right) dx dy + \int_0^{a_t} \int_0^{b_t} \left( [B_{ts}]^T [D^b_{ts}]^T [B_{ib}] \right) dx dy, \\
[K^e_{trs}] &= \int_0^{a_t} \int_0^{b_t} \left( [B_{rs}]^T \left( [D^b_{rs}] [B_{rs}^b] + [D^c_{rs}] [B_{rs}^c] + [D^p_{rs}] [B_{rs}^p] \right) dx dy \\
&\quad + [B_{rs}]^T [D^p_{rs}]^T [B_{rs}^p] \right) dx dy, \\
[K^e_{rsb}] &= \int_0^{a_t} \int_0^{b_t} \left( [B_{rs}^b]^T [D^b_{rs}] \right) dx dy, \\
[K^e_{tve}] &= \int_0^{a_t} \int_0^{b_t} \left( [B_{ve}]^T [D^v_{ve}] \right) dx dy, \\
[K^e_{tvp}] &= \int_0^{a_t} \int_0^{b_t} \left( [B_{vp}]^T [D^p_{vp}] \right) dx dy,
\end{align*}
\]

in which \(a_e\) and \(b_e\) are the length and the width of the element under consideration and the various rigidity matrices originated in the above elemental matrices are given in the Appendix. Substituting Eq. (23) and (24) into Eq. (21), the expression for kinetic energy \(T^e_{k}\) of the element can be obtained as

\[
T^e_{k} = \frac{1}{2} \{d^e\}^T [M^e] \{d^e\}
\]

in which

\[
M^e = \int_0^{a_t} \int_0^{b_t} \bar{m} \left[ N_i \right]^T \left[ N_i \right] dx dy
\]

and

\[
\bar{m} = 2\rho_e h_e + \rho_v h_v + \rho_p h_p + \sum_{k=1}^{N} \rho_t (h_{k+1} - h_k) + \sum_{k=1}^{N} \rho_b (h_{k+1} - h_k)
\]
Now, applying extended Hamilton’s principle for the non-conservative system [31], the following open loop governing equations of motion of an element are obtained:

\[
\begin{align*}
\left[ M^e \right] \left\{ \ddot{d}^e \right\} + \left[ K_n^e \right] \left\{ d^e \right\} + \left[ K_{tr}^e \right] \left\{ d^e \right\} + \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ d^e \right\} d\tau \\
+ \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ d^e \right\} d\tau = \left\{ F^e \right\} + \left\{ \{ F_{ip} \} + \{ F_{ftp} \} \right\} V
\end{align*}
\] (33)

\[
\begin{align*}
\left[ K_n^e \right] \left\{ d^e \right\} + \left[ K_{tr}^e \right] \left\{ d^e \right\} + \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ d^e \right\} d\tau + \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ d^e \right\} d\tau = \left\{ F^e \right\} V
\end{align*}
\] (34)

It should be noted here that for an element without integrated with the ACLD patch, the matrices \( \left\{ F_{ftp} \right\}, \left\{ F_{ip} \right\} \) and \( \left\{ F_{e} \right\} \) turn out to be the null matrices. The elemental governing equations are assembled into the global space to obtain the global equations of equilibrium as follows:

\[
\begin{align*}
\left[ M \right] \left\{ \ddot{X} \right\} + \left[ K_n \right] \left\{ X \right\} + \left[ K_{tr} \right] \left\{ X \right\} + \left[ K_{trsv} \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ X \right\} d\tau \\
+ \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ d^e \right\} d\tau = \{ F \} + \left\{ \{ F_{ip} \} + \{ F_{ftp} \} \right\} V
\end{align*}
\] (35)

\[
\begin{align*}
\left[ K_n \right] \left\{ X \right\} + \left[ K_{tr} \right] \left\{ X \right\} + \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ X \right\} d\tau + \left[ K_{trsv}^e \right] \int_0^1 G(t - \tau) \frac{\partial}{\partial \tau} \left\{ X \right\} d\tau = \left\{ F^e \right\} V
\end{align*}
\] (36)

where \( [M] \) is the global mass matrix; \( [K_n], [K_{tr}], [K_{trsv}] \) and \( [K_{trsv}] \) are the global stiffness matrices, \( \left\{ F_{ip} \right\}, \left\{ F_{ftp} \right\} \) and \( \left\{ F_{e} \right\} \) are the global electroelastic coupling vectors, \( \{ X \} \) and \( \{ X \} \) are the global nodal generalized displacement vectors, \( \{ F \} \) is the global nodal mechanical force vector. It may be noted that the elements of the matrices \( [K_n], [K_{tr}] \) and \( [K_{trsv}] \) are nonlinear functions of displacements. After invoking the boundary conditions and taking Laplace transform of Eqs. (35) and (36), the following global equations in Laplace domain are obtained:

\[
\begin{align*}
s^2 \left[ M \right] \left\{ \ddot{X} \right\} + \mathcal{L} \left( \left[ K_n \right] \left\{ X \right\} + \left[ K_{tr} \right] \left\{ X \right\} + \left[ K_{trsv} \right] s\tilde{G}(s) \left\{ \ddot{X} \right\} \right) \\
+ \mathcal{L} \left( \left[ K_{trsv}^e \right] s\tilde{G}(s) \left\{ \ddot{X} \right\} \right) = \left\{ \tilde{F} \right\} + \mathcal{L} \left( \left\{ \{ F_{ip} \} + \{ F_{ftp} \} \right\} \right) V
\end{align*}
\] (37)

\[
\begin{align*}
\mathcal{L} \left( \left[ K_{tr} \right] \left\{ X \right\} + \left[ K_{trsv} \right] s\tilde{G}(s) \left\{ \ddot{X} \right\} \right) + \left[ K_{trsv} \right] s\tilde{G}(s) \left\{ \ddot{X} \right\} + \left[ K_{trsv}^e \right] s\tilde{G}(s) \left\{ \ddot{X} \right\} = \left\{ \tilde{F} \right\} V
\end{align*}
\] (38)
where \( \{\bar{X}_t\}, \{\bar{X}_r\}, \{\bar{F}\} \) and \( \bar{V} \) are the Laplace transforms of \( \{X_t\}, \{X_r\}, \{F\} \) and \( V \), respectively.

Also, it may be mentioned that in the Laplace domain, the term \( sG(s) \) is referred to as a material modulus function \[49\]. Employing the GHM method for modeling the viscoelastic material in time domain, the material modulus function can be represented by a series of mini-oscillator terms as follows \[49\]:

\[
sG(s) = G^\infty \left[ 1 + \sum_{k=1}^{N} \alpha_k \frac{s^2 + 2\tilde{\xi}_k \omega_k s}{s^2 + 2\tilde{\xi}_k \omega_k s + \omega_k^2} \right]
\]

in which \( G^\infty \) corresponds to the equilibrium value of the modulus i.e. the final value of the relaxation \( G(t) \). Each mini-oscillator term is a second-order rational function involving three positive constants \( \alpha_k, \tilde{\xi}_k \) and \( \omega_k \). These constants govern the shape of the modulus function in the complex \( s \)-domain \[49\]. Now considering a GHM material modulus function with one mini-oscillator term \[49\] i.e.

\[
sG(s) = G^\infty \left[ 1 + \alpha \frac{s^2 + 2\tilde{\xi} \omega s}{s^2 + 2\tilde{\xi} \omega s + \omega^2} \right]
\]

the auxiliary dissipation coordinates \( Z, Z_r \) are introduced as follows \[60\]:

\[
sG(s)\{\bar{X}_t\} = G^\infty \left[ (1 + \alpha)\{\bar{X}_t\} - \alpha\{\bar{Z}_r\} \right], \quad sG(s)\{\bar{X}_r\} = G^\infty \left[ (1 + \alpha)\{\bar{X}_r\} - \alpha\{\bar{Z}_r\} \right] \tag{41}
\]

\[
\tilde{Z}(s) = \frac{\omega^2}{s^2 + 2\tilde{\xi} \omega s + \omega^2}\{\bar{X}_t\} \quad \text{and} \quad \tilde{Z}_r(s) = \frac{\omega^2}{s^2 + 2\tilde{\xi} \omega s + \omega^2}\{\bar{X}_r\} \tag{42}
\]

where \( \tilde{Z}(s) \) and \( \tilde{Z}_r(s) \) are the Laplace transforms of \( Z \) and \( Z_r \), respectively. Using Eqs. (40) and (42) in Eqs. (37) and (38), the open loop governing equations in Laplace domain are augmented as follows:

\[
s^2\{M\}\{\bar{X}_t\} + \{F\} + \{K_{tt}\}\{\bar{X}_t\} + \{K_{tr}\}\{\bar{X}_r\} + \{K^{v*}_{{\bar{X}_t}}\} G^\infty \left[ (1 + \alpha)\{\bar{X}_t\} - \alpha\{\bar{Z}_r\} \right] - \{K^{v*}_{{\bar{X}_r}}\} G^\infty \alpha\{\tilde{Z}(s)\}
\]

\[
+ \{K^{v*}_{{\bar{Z}_r}}\} G^\infty \left[ (1 + \alpha)\{\bar{X}_r\} - \alpha\{\bar{Z}_r\} \right] - \{K^{v*}_{{\bar{Z}_r}}\} G^\infty \alpha\{\tilde{Z}_r(s)\} = \{\tilde{F}\} + \{\tilde{F}_p\} \tilde{V} + \{F\}{\{F_p\}} \tilde{V} \tag{43}
\]

\[
\{F\} \left[ \{K_{tt}\}\{\bar{X}_t\} + \{K_{tr}\}\{\bar{X}_r\} + \{K^{v*}_{{\bar{X}_t}}\} G^\infty \left[ (1 + \alpha)\{\bar{X}_t\} - \alpha\{\bar{Z}_r\} \right] - \{K^{v*}_{{\bar{X}_r}}\} G^\infty \alpha\{\tilde{Z}_r(s)\}
\]

\[
+ \{K^{v*}_{{\bar{Z}_r}}\} G^\infty \left[ (1 + \alpha)\{\bar{X}_r\} - \alpha\{\bar{Z}_r\} \right] - \{K^{v*}_{{\bar{Z}_r}}\} G^\infty \alpha\{\tilde{Z}_r(s)\} = \{\tilde{F}\} + \{\tilde{F}_p\} \tilde{V} \tag{44}
\]
Taking inverse Laplace transforms of Eqs. (42) - (44) and condensing the global degrees of freedom \{X_r\} from the resulting equations in the time domain, the following equations are obtained:

\[
\begin{bmatrix} M \end{bmatrix}\begin{Bmatrix} \dot{X}_t \end{Bmatrix} + \begin{Bmatrix} K_x \end{Bmatrix}\begin{Bmatrix} X_t \end{Bmatrix} + \begin{Bmatrix} K_z \end{Bmatrix}\begin{Bmatrix} Z \end{Bmatrix} + \begin{Bmatrix} K_{zz} \end{Bmatrix}\begin{Bmatrix} Z_t \end{Bmatrix} = \begin{Bmatrix} F \end{Bmatrix} + \begin{Bmatrix} F_p \end{Bmatrix} \begin{Bmatrix} V \end{Bmatrix}
\]

(45)

\[
\begin{Bmatrix} \dot{Z}_t \end{Bmatrix} + 2\xi\dot{\omega}\begin{Bmatrix} \dot{Z} \end{Bmatrix} + \dot{\omega}^2\begin{Bmatrix} Z \end{Bmatrix} - \dot{\omega}^2\begin{Bmatrix} \dot{X}_t \end{Bmatrix} = 0
\]

(46)

\[
\begin{Bmatrix} \dot{Z}_r \end{Bmatrix} + 2\xi\dot{\omega}\begin{Bmatrix} \dot{Z}_r \end{Bmatrix} + \dot{\omega}^2\begin{Bmatrix} K_1 \end{Bmatrix}\begin{Bmatrix} X_t \end{Bmatrix} - \dot{\omega}^2\begin{Bmatrix} K_2 \end{Bmatrix}\begin{Bmatrix} Z \end{Bmatrix} - \dot{\omega}^2\begin{Bmatrix} K_3 \end{Bmatrix}\begin{Bmatrix} Z_r \end{Bmatrix} = \begin{Bmatrix} F_{pz} \end{Bmatrix} \begin{Bmatrix} V \end{Bmatrix}
\]

(47)

where

\[
\begin{Bmatrix} K_x \end{Bmatrix} = \begin{Bmatrix} K_{xt} \end{Bmatrix} - \begin{Bmatrix} K_{tr} \end{Bmatrix}\begin{Bmatrix} K_{rr} \end{Bmatrix}^{-1}\begin{Bmatrix} K_{rtr} \end{Bmatrix}, \quad \begin{Bmatrix} K_z \end{Bmatrix} = \begin{Bmatrix} K_{tr} \end{Bmatrix}\begin{Bmatrix} K_{rr} \end{Bmatrix}^{-1}\begin{Bmatrix} K_{trsv} \end{Bmatrix}^\top G^{\omega\alpha} - \begin{Bmatrix} K_{sv} \end{Bmatrix} G^{\omega\alpha},
\]

\[
\begin{Bmatrix} K_{zz} \end{Bmatrix} = \begin{Bmatrix} K_{tr} \end{Bmatrix}\begin{Bmatrix} K_{rr} \end{Bmatrix}^{-1}\begin{Bmatrix} K_{trsv} \end{Bmatrix}^\top G^{\omega\alpha} - \begin{Bmatrix} K_{sv} \end{Bmatrix} G^{\omega\alpha}, \quad \begin{Bmatrix} F_p \end{Bmatrix} = \begin{Bmatrix} F_p \end{Bmatrix} + \begin{Bmatrix} F_{pz} \end{Bmatrix} + \begin{Bmatrix} K_{tr} \end{Bmatrix}\begin{Bmatrix} K_{rr} \end{Bmatrix}^{-1}\begin{Bmatrix} F_p \end{Bmatrix},
\]

\[
\begin{Bmatrix} F \end{Bmatrix} = \begin{Bmatrix} F_t \end{Bmatrix} + \begin{Bmatrix} K_{tr} \end{Bmatrix}\begin{Bmatrix} K_{rr} \end{Bmatrix}^{-1}\begin{Bmatrix} F_t \end{Bmatrix}, \quad \begin{Bmatrix} K_1 \end{Bmatrix} = \begin{Bmatrix} K_{rtr} \end{Bmatrix}^{-1}\begin{Bmatrix} K_{rtr} \end{Bmatrix}, \quad \begin{Bmatrix} K_2 \end{Bmatrix} = \begin{Bmatrix} K_{rr} \end{Bmatrix}^{-1}\begin{Bmatrix} K_{trsv} \end{Bmatrix}^\top G^{\omega\alpha}
\]

Now, Eqs. (45) – (47) are combined to obtain the global open-loop equations of motion in the time domain as follows:

\[
\begin{bmatrix} M^* \end{bmatrix}\begin{Bmatrix} \ddot{X} \end{Bmatrix} + \begin{bmatrix} C^* \end{bmatrix}\begin{Bmatrix} \dot{X} \end{Bmatrix} + \begin{bmatrix} K^* \end{bmatrix}\begin{Bmatrix} X \end{Bmatrix} = \begin{bmatrix} F^* \end{bmatrix} + \begin{bmatrix} F_p^* \end{bmatrix} \begin{Bmatrix} V \end{Bmatrix}
\]

(49)

in which

\[
\begin{bmatrix} M^* \end{bmatrix} = \begin{bmatrix} M \end{bmatrix}, \quad \begin{bmatrix} C^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\xi\dot{\omega} & 0 \\ 0 & 0 & 2\xi\dot{\omega} \end{bmatrix}, \quad \begin{bmatrix} K^* \end{bmatrix} = \begin{bmatrix} K_x \end{bmatrix} - \dot{\omega}^2\begin{bmatrix} K_1 \end{bmatrix} - \dot{\omega}^2\begin{bmatrix} K_2 \end{bmatrix} - \dot{\omega}^2\begin{bmatrix} K_3 \end{bmatrix},
\]

\[
\begin{Bmatrix} F^* \end{Bmatrix} = \begin{Bmatrix} F_t \end{Bmatrix}, \quad \begin{Bmatrix} F_p^* \end{bmatrix} = \begin{Bmatrix} F_p \end{Bmatrix}, \quad \begin{Bmatrix} X \end{bmatrix} = \begin{Bmatrix} X_t \end{bmatrix} + \begin{Bmatrix} Z \end{bmatrix}, \quad \begin{Bmatrix} X_r \end{bmatrix} = \begin{Bmatrix} Z_r \end{bmatrix}
\]

(50)

5. Closed loop model
In order to apply the control voltage for activating the patches of the ACLD treatment, a simple velocity feedback control law has been employed. According to this law, the control voltage for each patch can be expressed in terms of the derivatives of the global nodal degrees of freedom as follows:

$$V^l = -K_d^l \dot{w} = -K_d^l \left[ U^l \right] \{ \dot{X} \}$$  \hspace{2cm} (51)

in which $K_d^l$ is the control gain for the $l$th patch and $\left[ U^l \right]$ is a unit vector defining the location of sensing the velocity signal that will be fed back to this patch. Finally, substituting Eq.(51) into Eq.(49), the equations of motion governing the closed loop dynamics of the substrate sandwich shells activated by the patch of the ACLD treatment can be obtained as follows:

$$\left[ M^* \right] \{ \ddot{X} \} + \left[ C_d^* \right] \{ \dot{X} \} + \left[ K^* \right] \{ X \} = \{ F^* \}$$  \hspace{2cm} (52)

where $\left[ C_d^* \right] = \left[ C^* \right] + \sum_{l=1}^{m} K_d^l \left[ F_p^* \right] \left[ U^l \right]$ is the active damping matrix and $m$ is the number of ACLD patches.

6. Results and Discussions

In this section, the numerical results are evaluated using the finite element model derived in the previous section for assessing the performance of the ACLD patches on controlling the geometrically nonlinear vibrations of doubly curved FFRC sandwich shells. FFRC facings with straight CNTs ($V_{CNT} \neq 0, \omega = 0$) and wavy CNTs ($V_{CNT} \neq 0, \omega \neq 0$) separated by a flexible HEREX honeycomb (soft) core integrated with centrally mounted patch of ACLD treatment (Fig. 4) are considered for evaluating the numerical results. The material properties of the core of the substrate sandwich shells are $E = 0.10363$ GPa, $G = 0.05$ GPa, $\nu = 0.33$ and $\rho_c = 130$ Kg/m$^3$ and those of the novel FFRC facings with different base fiber volume fraction are listed in Table 1 and are used for computing the numerical results. The above table displays the wide range of elastic properties of the novel FFRC with straight/wavy CNTs for three different base carbon fiber volume fractions ($V_f = 0.3, 0.5$ and $0.7$). The curvilinear length and the curvilinear width of the patch are 50% and 50% of the curvilinear length and the curvilinear width of the doubly curved sandwich shell.
curved sandwich shell, respectively. PZT-5 H/spur epoxy composite with 60% piezoelectric fiber volume fraction has been considered for the material of the constraining layer of the ACLD treatment. The elastic and the piezoelectric properties of this constraining layer are [28]:

\[
C^p_{11} = 9.29 \text{ GPa}, \ C^p_{12} = 6.18 \text{ GPa}, \ C^p_{13} = 6.05 \text{ GPa}, \ C^p_{33} = 35.44 \text{ GPa}, \ C^p_{23} = C^p_{13}, \ C^p_{55} = C^p_{44}, \ C^p_{44} = 1.58 \text{ GPa}, \ C^p_{11} = 1.54 \text{ GPa}, \ e_{31} = -0.1902 \text{ C/m}^2, \ e_{33} = 18.4107 \text{ C/m}^2
\]

The thicknesses of the constraining 1–3 PZC layer and the viscoelastic layer are considered to be 250 μm and 200 μm, respectively. Unless otherwise mentioned, the aspect ratio (a/H), the curvature ratio (R_1/a), the ratio of radii (R_2/R_1) and the thickness of the substrate sandwich shell are considered as 200, 5, 10 and 0.003 m, respectively while the thickness of each laminated orthotropic facing and that of the core are 0.001 m and 0.001 m, respectively. Also, unless otherwise mentioned each facing is a symmetric cross-ply (0°/90°/0°) FFRC laminate, the piezoelectric fiber orientation angle (ψ) in the PZC patch is considered to be 0° and the mechanical load (P) acting upward is assumed to be uniformly distributed. Considering a single term GHM expression, the values of α, ω and ξ are used as 11.42, 1.0261x 10^5 and 20, respectively [47]. The values of the shear modulus \(G^\infty\) and the density of the viscoelastic material (\(\rho_v\)) are used as 1.822x10^6 Pa and 1104 Kg/m^3, respectively [50]. The simply supported type one (SS1) boundary conditions at the edges of the overall plate considered for evaluating the numerical results are as follows

\[
v_0 = w_0 = \theta_y = \phi_y = \alpha_y = \beta_y = \gamma_y = \theta_z = \phi_z = \alpha_z = \beta_z = \gamma_z = 0, \text{ at } x=0, a
\]

and

\[
u_0 = w_0 = \theta_z = \phi_z = \alpha_z = \beta_z = \gamma_x = \theta_z = \phi_z = \alpha_z = \beta_z = \gamma_z = 0, \text{ at } y=0, b
\]

(53)

For the successful implementation of the GHM method for modelling the viscoelastic material layer of the ACLD treatment, the linear dynamic responses of a simply-supported symmetric cross-ply (0°/90°/0°/core/0°/90°/0°) doubly curved FFRC sandwich shell integrated with the patch of the ACLD treatment as shown in Fig. 4 are computed in the frequency domain using both the GHM method and the conventional complex modulus approach [50]. It may be noted that if the matrix \([B_4]\) appearing in the finite element formulation discussed in the previous
sections becomes a null matrix, Eq. (49) governing the open-loop dynamics becomes linear describing the linear behavior of the overall doubly curved FFRC sandwich shell. The frequency response functions using the GHM method are determined from the equation given below:

\[ \{X\} = \left( \left[ K^* + i\omega \left[ C^* - \omega^2 \left[ M^* \right] \right] \right)^{-1} \{F^*\} \right] \tag{54} \]

while, the same using conventional complex modulus approach are obtained by the equation:

\[ \{X_i\} = \left( \left[ \widetilde{K}(i\omega) - \omega^2 \left[ M^* \right] \right] \right)^{-1} \{F\} \tag{55} \]

where,

\[ \left[ \widetilde{K}(i\omega) \right] = \left( \left[ K_n^* \right] + \left[ K_{nv}^* \right] \right) G(i\omega) \]

\[ - \left( \left[ K_n^* \right] + \left[ K_{nv}^* \right] \right) G(i\omega) \times \left( \left[ K_{nr}^* \right] + \left[ K_{nnv}^* \right] \right) G(i\omega) \left( \left[ K_{nr}^* \right] + \left[ K_{nnv}^* \right] \right) G(i\omega) \right) \tag{56} \]

in which \( G(i\omega) \) is the complex shear modulus of the constrained viscoelastic layer. A pressure type of transverse pulse loading of 500 N/m² is applied on the plate to excite the first few modes and the linear frequency responses have been computed by both the approaches separately and are shown in Fig. 6. From this figure it can be observed that the frequency response obtained by both the approaches are almost overlapping with one another thus ascertaining the application of GHM method for modelling the constrained viscoelastic layer for accurately predicting the damping characteristics of the overall sandwich plates in the time domain. To verify the computation of the nonlinear stiffness matrices by the current FE model, the nonlinear fundamental frequency ratios of simply-supported laminated composite shell \( (0^0 / 90^0 / 0^0) \) studied by Panda and Singh [36] have been computed and are compared with the same obtained by Panda and Singh [36] as shown in Table 2. For such a comparison, the core of the sandwich shell is replaced by the conventional composite laminate thereby the sandwich shell resembles a laminated composite sandwich shell. From this Table it can be found that the two sets of results are in good agreement with each other ascertaining the validity of the current FE model. Furthermore, the present FE model has been used to compute the nonlinear fundamental frequency ratios of doubly curved FFRC sandwich shells with the top and the bottom facings being composed of FFRC laminate with straight CNTs \( (V_f = 0.5, \ V_{CNT} = 0.0269, \omega = 0, A/L_n = 0) \) and wavy CNTs \( (V_f = 0.5, \ V_{CNT} = 0.0573, \omega = 20\pi/L_n, A/L_n = 0.0450) \). These are tabulated in
Tables 3 and 4 which may be useful for other researchers for comparing their models. Having performed the validity of the present FE model, it has been employed to study the open-loop and the closed-loop dynamics of the doubly curved FFRC sandwich shells by computing the transverse deflection at the centre of the top surface of the shell \((a/2, b/2, H/2)\) To determine the magnitude of the distributed pulse loading which causes the plate to undergo large amplitude vibrations, backbone curves of the simply-supported doubly curved FFRC sandwich shells with different configurations are plotted as shown in Figs. 7-9. From these figures, one can determine the value of the amplitude ratio \((w/H)\) and the magnitude of the applied distributed pulse load which causes substantial nonlinearity in the dynamic characteristics of the overall shell. It may be observed from these figures that if the value of \(w/H\) greater than 0.2, the response of the overall doubly curved FFRC sandwich shells becomes nonlinear \((\omega_{NL}/\omega_l > 1)\). Hence, in order to deal with substantial nonlinearity \((\omega_{NL}/\omega_l > 1.2)\) the magnitude of the exciting distributed pulse load is selected in such a way that the maximum value of \(w/H\) for the uncontrolled response becomes greater than 1.8 which results in the value of fundamental frequency ratio \((\omega_{NL}/\omega_l)\) as greater than 1.2. It may also be observed from these figures that wavy CNTs causes more hardening and the sandwich shell with antisymmetric angle-ply FFRC facings exhibits more nonlinearity than the sandwich shell with cross-ply FFRC facings for the same amplitude ratio. The control voltage supplied is proportional to the velocity of the point \((a/2, b/2, H/2)\) on the centre of the doubly curved sandwich shell. The control gain \((K_d)\) is choosen arbitrarily such that the initial value of the control voltage \((V)\) supplied to attenuate the vibrations does not exceed a nominal value of 300 Volts. Keeping the above parameters in view, the nonlinear transient responses of simply-supported (SS1) symmetric cross-ply doubly curved FFRC sandwich shells with facings containing straight CNTs \((V_{CNT} \neq 0, \omega = 0)\) and wavy CNTs \((V_{CNT} \neq 0, \omega \neq 0)\) are illustrated in Figs. 10 and 11. The above figures displaying the transient decay responses corresponding to undamped (when the patch is inactive \(K_d = 0\)) and damped (when the patch is active \(K_d \neq 0\)) reveal that the active ACLD patch is more efficient over the inactive ACLD patch (Passive mode) for controlling the geometrically nonlinear vibrations of these doubly curved FFRC sandwich shells. The control voltages corresponding to the above responses are found to be quite nominal which are shown in Figs. 12 and 13. Since the control
voltage is directly proportional to the velocity of a point on the shell, the decay of control voltages with time shown in Figs. 12 and 13 imply that the shell velocity at any point also decays with the time owing to the stability of the overall shell. The phase plots presented in Figs. 14 and 15 also establish the observation indicating the stability of the shell. The transient controlled responses of simply-supported (SS1) doubly curved FFRC sandwich shell with FFRC facings containing zero CNTs \( (V_{\text{CNT}} = 0, \text{base composite}) \), straight CNTs 
\( (V_f = 0.5, V_{\text{CNT}} = 0.0269, \omega = 0, A/L_n = 0) \) and wavy CNTs 
\( (V_f = 0.5, V_{\text{CNT}} = 0.0573, \omega = 20\pi/L_n, A/L_n = 0.0450) \) for the same load and maximum control voltage are compared in Fig. 16. Displayed in this figure is also the response of the shell without CNTs \( (V_{\text{CNT}} = 0) \) subjected to same load and maximum control voltage. It can be observed from this figure that if the carbon fibers of the facings of the sandwich shell are coated with radially grown CNTs, the performance of the ACLD patch improves over that without CNTs. Also, waviness of the CNTs causes further improvement of the performance of the ACLD patch. Next the transient responses of simply-supported (SS1) doubly curved sandwich shell with FFRC facings containing radially grown wavy CNTs \( (\omega = 20\pi/L_n) \) for different base carbon fiber volume fractions are shown in Fig. 17. The responses shown in this figure exhorts that the amplitude ratio and the decay times decrease with the increase in the base carbon fiber volume fraction for the same value of the transverse distributed pulse loading and the maximum control voltage \( \text{max}(V) = 300 \text{ Volts} \).

The effect of transverse actuation by the constraining layer of the ACLD patch on the damping characteristics of the overall sandwich shell has been studied by plotting the damped responses of the simply-supported doubly curved sandwich shell with FFRC facings containing wavy CNTs 
\( (V_f = 0.5, V_{\text{CNT}} = 0.0573, \omega = 20\pi/L_n, A/L_n = 0.0450) \) for the cases when the value of the piezoelectric coefficient \( e_{33} \) is zero or non zero as shown in Fig. 18. From this figure it is evident that the contribution of the transverse actuation by the active PZC constraining layer is significantly larger than that of the in-plane actuation by the same for suppressing the geometrically nonlinear vibrations of the doubly curved FFRC substrate sandwich shells.

Next, the effect of variation of the piezoelectric fiber orientation angle in the 1-3 PZC constraining layer on the performance of ACLD patches for controlling the geometrically
nonlinear vibrations of FFRC sandwich plates has been investigated. For such a study, a performance index \( I_d \) is defined as follows:

\[
I_d = \frac{w(\frac{a}{2}, \frac{b}{2}, \frac{H}{2}, 0) - w(\frac{a}{2}, \frac{b}{2}, \frac{H}{2}, 0.3)}{w(\frac{a}{2}, \frac{b}{2}, \frac{H}{2}, 0)} \times 100
\]

(62)

where, \( I_d \) is measure of percentage diminution of the displacement at the point \((a/2, b/2, H/2)\) of the controlled response of the doubly curved FFRC sandwich shell undergoing geometrically nonlinear transient vibrations after 0.3 second. By varying the piezoelectric fiber orientation angle \( \psi \) of the 1-3 PZC layer with respect to \( z \)-axis in the \( xz \)-plane, the corresponding values of performance index \( I_d \) have been determined. For a same value of maximum control voltage \( \text{max}(V) = 300 \text{ Volts} \) Figs. 19-21 display the variation of the performance index \( (I_d) \) with the piezoelectric fiber orientation angle \( \psi \) of the simply-supported sandwich shells with different configurations of FFRC facings. It may be observed from these figures that the performance index \( (I_d) \) attains a maximum value when the value of the piezoelectric fiber orientation angle \( \psi \) is 45° in the \( xz \)-plane. Similar results are also found when the piezoelectric fiber orientation angle is varied in the \( yz \)-plane. However, for the sake of brevity they are not presented here. It may be further observed from Fig. 19 that the radially grown CNTs cause improvement of the performance index while the control authority of the ACLD patch is more in the case of the FFRC facings containing wavy CNTs than that in the of FFRC facings with straight CNTs. This is attributed to the fact the wavy CNTs considered here improve the effective axial elastic coefficients of the FFRC layers. For a particular value of wave frequency \( \omega \) of the wavy CNTs, the control authority of the ACLD patches increases with the increase in the CNT volume fraction (Fig. 20). Also, the performance of the ACLD patches is better in case of the sandwich plate with antisymmetric angle-ply FFRC facings than that in case of sandwich plates with cross-ply facings for controlling their geometrically nonlinear vibrations (Fig. 21).

7. Conclusions

In this paper, active constrained layer damping of geometrically nonlinear vibrations of
doubly curved sandwich shells with their faces being composed of FFRC has been studied. The constructional feature of FFRC is that the straight or wavy CNTs are radially grown on the carbon fibers of the base composite. The plane of the waviness of the wavy CNTs is coplanar with the plane of carbon fiber. The constraining layer of the ACLD patch is considered to be made of the vertically/obliquely reinforced 1-3 PZC. Based on the layerwise displacement theories and the Von Kármán type geometrically non-linear strain-displacement relations, a three dimensional nonlinear electro-mechanical finite element model of the overall doubly curved sandwich shell has been derived. The viscoelastic layer of the ACLD patch is modeled by using the GHM approach, which is a time domain formulation. A simple velocity feedback control law is used to introduce active damping. Various substrate doubly curved sandwich shells composed of symmetric and antisymmetric cross-ply and anti symmetric angle-ply FFRC facings and a flexible HEREX (C70 130) honeycomb core are considered for evaluation of the numerical results. The geometrically nonlinear dynamics of the doubly curved FFRC substrate sandwich shells are of hardening type. The numerical results for active responses reveal that the ACLD treatment significantly improves the active damping characteristics of the simply-supported doubly curved FFRC sandwich shells over the passive damping for suppressing the geometrically nonlinear transient vibrations of the shells. The performance of the ACLD patch for attenuating the geometrically nonlinear vibrations of doubly curved sandwich shells with laminated carbon fiber reinforced composite facings significantly increases if the carbon fibers are coated with radially grown CNTs. Wavy CNTs being coplanar with the plane of the fiber cause more improvement of the performance of the ACLD patch than the straight CNTs. For active damping of geometrically nonlinear vibrations of doubly curved FFRC sandwich shells, the contribution of vertical actuation by the vertically reinforced 1-3 PZC layer is significantly larger than that of the in-plane actuation by the same. The performance of the patch becomes maximum when the piezoelectric fibers are obliquely aligned (ψ = 45°) in the constraining layer of the ACLD patch. Furthermore, the controllability of the ACLD patch for active damping of large amplitude vibrations of doubly curved sandwich shells with antisymmetric angle-ply FFRC facings is more than that in the case of cross-ply FFRC facings. Based on the research carried out here, it may be concluded that even the wavy CNTs can be properly exploited for achieving structural benefits from the excellent elastic properties of CNTs and very light-weight and efficient smart structures can be developed in future using wavy CNTs.
Table 1. Material properties of the base composite and the FFRC

<table>
<thead>
<tr>
<th>Carbon fiber volume fraction ($V_f$)</th>
<th>CNT Volume Fraction and CNT wave frequency</th>
<th>$C_{11}$ (GPa)</th>
<th>$C_{12}$ (GPa)</th>
<th>$C_{13}$ (GPa)</th>
<th>$C_{22}$ (GPa)</th>
<th>$C_{23}$ (GPa)</th>
<th>$C_{33}$ (GPa)</th>
<th>$C_{44}$ (GPa)</th>
<th>$C_{55}$ (GPa)</th>
<th>$C_{66}$ (GPa)</th>
<th>$\rho$ (Kg/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>$V_{CNT} = 0$</td>
<td>76.9063</td>
<td>6.7112</td>
<td>6.7112</td>
<td>11.0328</td>
<td>7.1056</td>
<td>11.0328</td>
<td>1.9636</td>
<td>2.0891</td>
<td>2.0891</td>
<td>1319.0</td>
</tr>
<tr>
<td>Straight CNT</td>
<td>$V_{CNT} = 0.0344, \omega = 0, A/L_a = 0$</td>
<td>78.0415</td>
<td>7.9395</td>
<td>7.9395</td>
<td>26.1586</td>
<td>13.3566</td>
<td>26.1586</td>
<td>6.4010</td>
<td>2.4475</td>
<td>2.4475</td>
<td>1326.0</td>
</tr>
<tr>
<td>wavy CNT</td>
<td>$V_{CNT} = 0.0380, \omega = 10\pi/L_a, A/L_a = 0.0211$</td>
<td>80.0387</td>
<td>10.4824</td>
<td>10.4824</td>
<td>24.3655</td>
<td>12.4524</td>
<td>24.3655</td>
<td>5.9566</td>
<td>6.4267</td>
<td>6.4267</td>
<td>1326.7</td>
</tr>
<tr>
<td></td>
<td>$V_{CNT} = 0.0465, \omega = 20\pi/L_a, A/L_a = 0.0211$</td>
<td>90.6505</td>
<td>13.1748</td>
<td>13.1748</td>
<td>22.2527</td>
<td>11.4948</td>
<td>22.2527</td>
<td>5.3789</td>
<td>8.0980</td>
<td>8.0980</td>
<td>1328.4</td>
</tr>
<tr>
<td></td>
<td>$V_{CNT} = 0.0576, \omega = 30\pi/L_a, A/L_a = 0.0211$</td>
<td>106.9082</td>
<td>14.3654</td>
<td>14.3654</td>
<td>21.3422</td>
<td>11.1728</td>
<td>21.3422</td>
<td>5.0847</td>
<td>8.5241</td>
<td>8.5241</td>
<td>1330.7</td>
</tr>
<tr>
<td>0.5</td>
<td>$V_{CNT} = 0$</td>
<td>122.2642</td>
<td>7.3774</td>
<td>7.3774</td>
<td>13.0124</td>
<td>8.0660</td>
<td>13.0124</td>
<td>2.4732</td>
<td>2.8302</td>
<td>2.8302</td>
<td>1445.0</td>
</tr>
<tr>
<td>Straight CNT</td>
<td>$V_{CNT} = 0.0269, \omega = 0, A/L_a = 0$</td>
<td>123.2849</td>
<td>8.7170</td>
<td>8.7170</td>
<td>24.8835</td>
<td>12.6299</td>
<td>24.8835</td>
<td>6.1268</td>
<td>3.3501</td>
<td>3.3501</td>
<td>1450.5</td>
</tr>
<tr>
<td>wavy CNT</td>
<td>$V_{CNT} = 0.0375, \omega = 10\pi/L_a, A/L_a = 0.0450$</td>
<td>134.2089</td>
<td>12.5775</td>
<td>12.5775</td>
<td>22.4455</td>
<td>11.4623</td>
<td>22.4455</td>
<td>5.4916</td>
<td>9.0809</td>
<td>9.0809</td>
<td>1452.6</td>
</tr>
<tr>
<td></td>
<td>$V_{CNT} = 0.0573, \omega = 20\pi/L_a, A/L_a = 0.0450$</td>
<td>163.2750</td>
<td>13.8126</td>
<td>13.8126</td>
<td>21.9361</td>
<td>11.3916</td>
<td>21.9361</td>
<td>5.2723</td>
<td>9.4603</td>
<td>9.4603</td>
<td>1456.6</td>
</tr>
<tr>
<td></td>
<td>$V_{CNT} = 0.0794, \omega = 30\pi/L_a, A/L_a = 0.0450$</td>
<td>195.7687</td>
<td>14.1434</td>
<td>14.1434</td>
<td>22.7349</td>
<td>11.9347</td>
<td>22.7349</td>
<td>5.4001</td>
<td>9.6056</td>
<td>9.6056</td>
<td>1461.1</td>
</tr>
<tr>
<td>0.7</td>
<td>$V_{CNT} = 0$</td>
<td>167.7390</td>
<td>8.3014</td>
<td>8.3014</td>
<td>15.9179</td>
<td>9.2379</td>
<td>15.9179</td>
<td>3.3400</td>
<td>4.3860</td>
<td>4.3860</td>
<td>1571.0</td>
</tr>
<tr>
<td>Straight CNT</td>
<td>$V_{CNT} = 0.0150, \omega = 0, A/L_a = 0$</td>
<td>168.4045</td>
<td>9.2890</td>
<td>9.2890</td>
<td>22.9629</td>
<td>11.5712</td>
<td>22.9629</td>
<td>5.6958</td>
<td>5.0660</td>
<td>5.0660</td>
<td>1574.1</td>
</tr>
<tr>
<td>wavy CNT</td>
<td>$V_{CNT} = 0.0382, \omega = 10\pi/L_a, A/L_a = 0.1129$</td>
<td>199.4109</td>
<td>11.9698</td>
<td>11.9698</td>
<td>21.8725</td>
<td>11.1762</td>
<td>21.8725</td>
<td>5.3482</td>
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<td>1578.7</td>
</tr>
<tr>
<td></td>
<td>$V_{CNT} = 0.0705, \omega = 20\pi/L_a, A/L_a = 0.1129$</td>
<td>240.5038</td>
<td>12.4910</td>
<td>12.4910</td>
<td>24.9625</td>
<td>12.0714</td>
<td>24.9625</td>
<td>6.4456</td>
<td>12.9435</td>
<td>12.9435</td>
<td>1585.3</td>
</tr>
<tr>
<td></td>
<td>$V_{CNT} = 0.1037, \omega = 30\pi/L_a, A/L_a = 0.1129$</td>
<td>292.1637</td>
<td>11.5644</td>
<td>11.5644</td>
<td>32.5068</td>
<td>26.0236</td>
<td>32.5068</td>
<td>3.2416</td>
<td>9.5223</td>
<td>9.5223</td>
<td>1592.1</td>
</tr>
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</table>
Table-2 Nonlinear fundamental frequency ratio ($\omega_{NL}/\omega_L$) of simply supported (SS1) laminated spherical shell ($0^\circ/90^\circ/0^\circ$) for different curvature ratios and amplitude ratios with $a/b=1$, $a/H=10$ and $R_1=R_2=R$.

<table>
<thead>
<tr>
<th>Curvature ratios ($R/a$)</th>
<th>($W_{\text{max}}/H$)</th>
<th>Panda and Singh. [36]</th>
<th>Present FE soln.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.6</td>
<td>1.0904</td>
<td>1.0896</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>1.1610</td>
<td>1.1608</td>
</tr>
<tr>
<td>50</td>
<td>1.4</td>
<td>1.2532</td>
<td>1.2524</td>
</tr>
<tr>
<td>100</td>
<td>1.8</td>
<td>1.4143</td>
<td>1.3986</td>
</tr>
</tbody>
</table>

Table 3. Non-linear fundamental frequency ratios ($\omega_{NL}/\omega_L$) of simply-supported doubly curved FFRC sandwich shells with straight CNT FFRC laminate facings

$(V_f = 0.5, \ V_{\text{CNT}} = 0.0269, \omega = 0, A/L_n = 0), (a/b=1, h_c/h=1, R_2/R_1 = 10)$

<table>
<thead>
<tr>
<th>Type of Lamina</th>
<th>$\omega_{NL}/\omega_L$</th>
</tr>
</thead>
</table>
| $(0^\circ/90^\circ/0^\circ$ | \begin{tabular}{c|ccccccc}
| w/H | 0.2 | 0.4 | 0.8 | 1.0 | 1.4 | 1.6 \\
| a/H |     |     |     |     |     |     |
| 20  | 1.0121 | 1.0435 | 1.1554 | 1.2307 | 1.4074 | 1.5049 |
| 40  | 1.0079 | 1.0265 | 1.0550 | 1.1382 | 1.2491 | 1.3123 |
| 80  | 1.0079 | 1.0230 | 1.0774 | 1.1141 | 1.2038 | 1.3109 |
| 100 | 1.0079 | 1.0231 | 1.0773 | 1.1132 | 1.2007 | 1.2511 |
| $(45^\circ/0^\circ/45^\circ$ | \begin{tabular}{c|ccccccc}
| w/H | 0.2 | 0.4 | 0.8 | 1.0 | 1.4 | 1.6 \\
| a/H |     |     |     |     |     |     |
| 20  | 1.0123 | 1.0442 | 1.1577 | 1.2340 | 1.4130 | 1.5118 |
| 40  | 1.0079 | 1.0262 | 1.0917 | 1.1369 | 1.2469 | 1.3097 |
| 80  | 1.0078 | 1.0235 | 1.0766 | 1.1129 | 1.2017 | 1.2528 |
| 100 | 1.0084 | 1.0243 | 1.0776 | 1.1121 | 1.1987 | 1.2485 |
| $(−45^\circ/45^\circ/−45^\circ/45^\circ$ | \begin{tabular}{c|ccccccc}
| w/H | 0.2 | 0.4 | 0.8 | 1.0 | 1.4 | 1.6 \\
| a/H |     |     |     |     |     |     |
| 20  | 1.0322 | 1.1002 | 1.3146 | 1.4472 | 1.7427 | 1.4018 |
| 40  | 1.0209 | 1.0598 | 1.1822 | 1.2605 | 1.4406 | 1.5395 |
| 80  | 1.0205 | 1.0528 | 1.1477 | 1.2075 | 1.3457 | 1.4219 |
| 100 | 1.0221 | 1.551 | 1.1489 | 1.2074 | 1.3416 | 1.4155 |
Table 4. Non-linear fundamental frequency ratios ($\omega_{NL}/\omega_L$) of simply-supported doubly curved FFRC sandwich shells with wavy CNT FFRC laminate facings

$(V_f = 0.5, \ V_{CNT} = 0.0269, \omega = 20\pi/L_n, A/L_n = 0.0450), \ (a/b = 1, 2h_c/h = 1, R_2/R_1 = 10)$

<table>
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<tr>
<th>Type of Lamina</th>
<th>$\omega_{NL}/\omega_L$</th>
<th>$\omega_{NL}/\omega_L$</th>
<th>$\omega_{NL}/\omega_L$</th>
<th>$\omega_{NL}/\omega_L$</th>
<th>$\omega_{NL}/\omega_L$</th>
<th>$\omega_{NL}/\omega_L$</th>
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<td>0.8</td>
<td>1.0</td>
<td>1.4</td>
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<tr>
<td>/core/ $0^\circ/90^\circ/0^\circ$ )</td>
<td>$a/H$</td>
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<td>$w/H$</td>
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<tr>
<td>$(45^\circ/45^\circ/45^\circ$</td>
<td>$w/H$</td>
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<td>0.4</td>
<td>0.8</td>
<td>1.0</td>
<td>1.4</td>
</tr>
<tr>
<td>/core/ $45^\circ/45^\circ/45^\circ$ )</td>
<td>$a/H$</td>
<td></td>
<td></td>
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References


Fig. 1 schematic diagram of (a) laminae of vertically ($\psi = 0$) /obliquely ($\psi \neq 0$) reinforced 1-3 PZC (b) a lamina of the continuous FFRC containing either straight CNTs or wavy CNTs being coplanar with the longitudinal plane of the carbon fiber.

Fig. 2 schematic diagrams of (a) the fuzzy fiber coated with straight CNTs (b) the fuzzy fiber coated with wavy CNTs being coplanar with the longitudinal plane of the carbon fiber.
Fig. 3 Representative volume element of the CNT-reinforced nanocomposite containing a wavy CNT radially grown on the carbon fiber and coplanar with 1-3 plane.

Fig. 4 Schematic diagram of smart doubly curved FFRC sandwich shell integrated with the ACLD Patch.
Fig. 5 Kinematics of deformation of any transverse cross section of the smart doubly curved FFRC sandwich shell integrated with the ACLD patch (a) $xz$ – plane (b) $yz$ – plane
Fig. 6 Linear frequency responses for central deflection of a simply-supported doubly curved FFRC sandwich shell \((0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)\) obtained by the GHM method and the complex modulus approach \((V_f = 0.5, \; V_{\text{CNT}} = 0.0269, \; \omega = 20\pi/L_n, \; A/L_n = 0.0450, R_2/R_1 = 10)\)

Fig. 7 Backbone curves of a simply-supported doubly curved FFRC sandwich shell \((0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)\) FFRC facings with wavy CNTs \((\omega = 20\pi/L_n)\).
Fig. 8 Backbone curves of a simply-supported doubly curved sandwich shell

\( (0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ) \) with FFRC facings

\((V_f = 0.5, \ V_{CNT} = 0.0269, \ \omega = 20\pi/L_n, \ A/L_n = 0.0450)\).

Fig. 9 Backbone curves of a simply-supported doubly curved sandwich shell

\( (0^\circ/90^\circ/0^\circ/\text{core}/0^\circ/90^\circ/0^\circ) \) with FFRC facings \((V_f = 0.5)\).
Fig. 10 Non-linear transient responses of a simply-supported simply-supported doubly curved sandwich shell \((0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)\) undergoing ACLD with FFRC facings containing straight CNTs \((V_f = 0.5, V_{CNT} = 0.0269, \omega = 0, A/L_n = 0)\).

Fig. 11 Non-linear transient responses of a simply-supported simply-supported doubly curved sandwich shell \((0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)\) undergoing ACLD with FFRC facings containing wavy CNTs \((V_f = 0.5, V_{CNT} = 0.0269, \omega = 20\pi/L_n, A/L_n = 0.0450)\).
Fig. 12 Control voltages required for the ACLD of non-linear transient vibrations of the simply-supported doubly curved sandwich shell \((0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)\) with FFRC facings containing straight CNTs \((V_f = 0.5, \; V_{\text{CNT}} = 0.0269, \; \omega = 0, \; A/L_n = 0)\).

Fig. 13 Control voltages required for the ACLD of non-linear transient vibrations of the simply-supported doubly curved sandwich shell \((0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)\) with FFRC facings containing wavy CNTs \((V_f = 0.5, \; V_{\text{CNT}} = 0.0269, \; \omega = 20\pi/L_n, \; A/L_n = 0.0450)\).
**Fig. 14** Phase plot of the simply-supported doubly curved sandwich shell 
$(0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)$ with FFRC facings ($V_f = 0.5$, $V_{CNT} = 0.0269$, $\omega = 0$, $A/L_n = 0$) when the ACLD patch control the non-linear vibrations.

**Fig. 15** Phase plot of the simply-supported doubly curved sandwich shell 
$(0^\circ / 90^\circ / 0^\circ / \text{core} / 0^\circ / 90^\circ / 0^\circ)$ with FFRC facings ($V_f = 0.5$, $V_{CNT} = 0.0269$, $\omega = 20\pi/L_n$, $A/L_n = 0.0450$) with FFRC facings when the ACLD patch control the non-linear vibrations.
Fig. 16 Effect of waviness on the ACLD of geometrically non-linear vibrations of a simply-supported doubly curved sandwich shell \((0^\circ/90^\circ/0^\circ/core/0^\circ/90^\circ/0^\circ)\) with FFRC facings \((V_f = 0.5, P = 400 \text{ N/m}^2, \max(V) = 300 \text{ Volts})\).

Fig. 17 Effect of volume fraction of carbon fiber on the ACLD of geometrically non-linear vibrations of a simply-supported doubly curved sandwich shell \((0^\circ/90^\circ/0^\circ/core/0^\circ/90^\circ/0^\circ)\) with FFRC facings containing wavy CNTs \((\omega = 20\pi/L_n, A/L_n = 0.0450)\).
**Fig. 18** Contribution of in-plane and transverse actuations by the 1-3 PZC constraining layer in the controlled response of simply-supported doubly curved sandwich shell (0°/90°/0°/core/0°/90°/0°) with FFRC facings ($V_f = 0.5$, $V_{CNT} = 0.0269$, $\omega = 0$, $A/L_n = 0$) undergoing geometrically nonlinear vibrations ($max(V) = 300$ Volts).

**Fig. 19** Effect of variation of piezoelectric fiber orientation angle ($\psi$) in the 1-3 PZC constraining layer of the ACLD patch when the fibers are coplanar with the $xz$-plane for the simply-supported doubly curved sandwich shell (0°/90°/0°/core/0°/90°/0°) with FFRC facings ($V_f = 0.5$, $max(V) = 300$ Volts).
Fig. 20 Effect of variation of piezoelectric fiber orientation angle ($\psi$) in the 1-3 PZC constraining layer of the ACLD patch when the fibers are coplanar with the $xz$-plane for the simply-supported doubly curved sandwich shell ($0^\circ/90^\circ/0^\circ$/core/$0^\circ/90^\circ/0^\circ$) with FFRC facings containing wavy CNTs ($\omega = 20\pi/L_n$, $\text{max}(V) = 300$ Volts).

Fig. 21 Effect of variation of piezoelectric fiber orientation angle ($\psi$) in the 1-3 PZC constraining layer of the ACLD patch when the fibers are coplanar with the $xz$-plane for the simply-supported doubly curved sandwich shell ($0^\circ/90^\circ/0^\circ$/core/$0^\circ/90^\circ/0^\circ$) with FFRC facings ($V_f = 0.5$, $V_{\text{CNT}} = 0.0269$, $\omega = 20\pi/L_n$, $A/L_n = 0.0450$, $\text{max}(V) = 300$ Volts).
Appendix:
In Eq. (25), the matrices \([Z_i]\), \([Z_2]\), \([Z_3]\), \([Z_4]\), \([Z_5]\), \([Z_6]\), \([Z_7]\) and \([Z_8]\) are given by
\[
[Z_i] = \begin{bmatrix}
Z_i & 0 & 0 & 0 \\
0 & Z_i & 0 & 0 \\
0 & 0 & h_c & 0 \\
0 & 0 & 0 & h_c
\end{bmatrix}, \quad [Z_2] = \begin{bmatrix}
(-h_c)I & 0 & 0 \\
0 & (-h_c)I & 0 \\
0 & 0 & -h_c \\
0 & 0 & 0
\end{bmatrix}, \quad [Z_3] = \begin{bmatrix}
(h_c)I & 0 & 0 \\
0 & (h_c)I & 0 \\
0 & 0 & h_c \\
0 & 0 & 0
\end{bmatrix}, \quad [Z_4] = \begin{bmatrix}
(h_c) & 0 & 0 \\
0 & (h_c) & 0 \\
0 & 0 & h_v \\
0 & 0 & 0
\end{bmatrix}, \quad [Z_5] = \begin{bmatrix}
(0) & 0 & 0 \\
0 & (0) & 0 \\
0 & 0 & h_v \\
0 & 0 & 0
\end{bmatrix}, \quad [Z_6] = \begin{bmatrix}
(0) & 0 & 0 \\
0 & (0) & 0 \\
0 & 0 & h_v \\
0 & 0 & 0
\end{bmatrix}, \quad [Z_7] = \begin{bmatrix}
(0) & 0 & 0 \\
0 & (0) & 0 \\
0 & 0 & h_v \\
0 & 0 & 0
\end{bmatrix}, \quad [Z_8] = \begin{bmatrix}
(0) & 0 & 0 \\
0 & (0) & 0 \\
0 & 0 & h_v \\
0 & 0 & 0
\end{bmatrix}
\]
where
The various submatrices $[B_{thi}]$, $[B_{rbi}]$, $[B_{tsi}]$ and $[B_{rsi}]$ appearing in Eq. (26)
The various rigidity matrices and the rigidity vectors for electro-elastic coupling elemental matrices appearing in Eqs. (29) and (30) are given by

\[
\begin{align*}
\{D^b\} &= \sum_{k=1}^{N} \int [\bar{C}^b] \bar{k} \, dz, \\
\{D^c\} &= \sum_{k=1}^{N} \int [\bar{C}^c] \bar{k} \, dz, \\
\{D^d\} &= \sum_{k=1}^{N} \int [\bar{C}^d] \bar{k} \, dz,
\end{align*}
\]

\[
\begin{align*}
\{D_{bb}\} &= \sum_{k=1}^{N} \int [\bar{C}_{bb}^b] \bar{k} \, dz, \\
\{D_{cc}\} &= \sum_{k=1}^{N} \int [\bar{C}_{cc}^c] \bar{k} \, dz, \\
\{D_{dd}\} &= \sum_{k=1}^{N} \int [\bar{C}_{dd}^d] \bar{k} \, dz.
\end{align*}
\]